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THIS IS VOLUME 2 OF A THREE-VOLUME EXPERIMENTAL EDITION CONTAINING A SEQUENCE OF ENRICHED MATERIALS FOR SEVENTH-GRADE MATHEMATICS. THESE MATERIALS CAN BE USED EITHER FOR A PROGRAM OF INDIVIDUALIZED INSTRUCTION FOR THE ACCELERATED STUDENT OR FOR CLASSROOM PRESENTATION BY THE TEACHER. THE PRESENTATION OF THE MATERIAL IS SUCH AS TO REFLECT CHANGES IN CONTENT, TECHNIQUE, APPROACH AND EMPHASIS. INSTRUCTIONAL UNITS ON A NUMBER OF SEQUENTIALLY RELATED TOPICS ARE DESIGNED TO INCORPORATE MODERN TERMINOLOGY WITH THE TRADITIONAL TOPICS AND TO INTRODUCE NEW CONCEPTS AS APPROPRIATE. THIS VOLUME INCLUDES MATERIALS FOR (1) MULTIPLICATION OF INTEGERS, (2) LATTICE POINTS IN THE PLANE AND MAPPING ON Z X Z, AND (3) SETS AND RELATIONS. (RP)

MATHEMATICS I

(Experimental Edition)

Volume 2

Secondary School Mathematics
Curriculum Improvement Study

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Chapter

CHAPTER 6 MULTIPLICATION OF INTEGERS

6.1 Operational Systems (W, ·) and (Z, ·)

In Chapter 4 we learned how to add and subtract integers. It is natural to ask how integers should be multiplied.

With respect to the operation of addition, the whole numbers are isomorphic to the positive integers. That is, addition of whole numbers is just like addition of positive integers. It is reasonable to require that multiplication preserve this close relationship between the whole numbers and positive integers. Let us recall some of the properties of (W, ·) which we should like to carry over into (Z, ·).

1. For all whole numbers a and b, $a \cdot b = b \cdot a$.
   (Commutative Property of Multiplication)
   For example: $3 \cdot 7 = 7 \cdot 3$
2. For all whole numbers a, b, and c, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
   (Associative Property of Multiplication)
   For example: $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$
3. For every whole number a, $1 \cdot a = a \cdot 1 = a$.
   (1 is a Multiplicative Identity in W)
   For example: $1 \cdot 7 = 7 \cdot 1 = 7$
4. For every whole number a, $a \cdot 0 = 0 \cdot a = 0$.
   (Multiplication Property of Zero)
   For example: $3 \cdot 0 = 0 \cdot 3 = 0$
5. For all whole numbers a, b, and c, if $c \neq 0$ and $c \cdot a = c \cdot b$, then $a = b$.
   (Cancellation Property of Multiplication)
   For example· If $7 \cdot a = 7 \cdot 13$ then $a = 13$

There is one property of the operational system (W, +, ·) which relates the operations of addition and multiplication. This property is illustrated in the following example.

Suppose we compute the product 7×13 in the usual way:

$$\begin{array}{r} 13 \\ \underline{\times 7} \\ 91 \end{array}$$

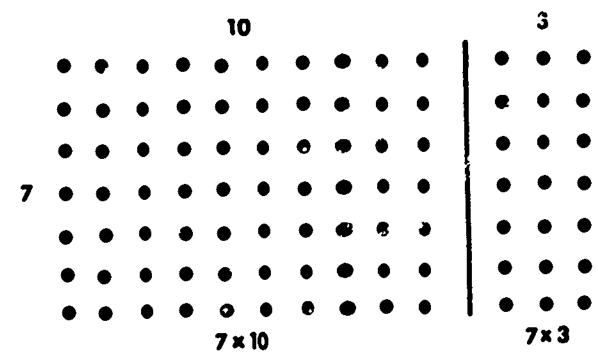
In this computation, we have actually computed two simpler products

$$7\times3 = 21 \quad \text{and} \quad 7\times10 = 70$$

and then computed their sum

$$21+70 = 91$$

The reason this works is easy to understand if we picture the product 7×13 as a rectangular array that has been split into two arrays:



On the left we have a 7×10 array and on the right a 7×3 array. The number of elements in the array does not change by the splitting, so we have

$$7 \cdot (10 + 3) = 7 \cdot 10 + 7 \cdot 3$$

Similarly, we know that

$$7 \cdot (4 + 6) = 7 \cdot 4 + 7 \cdot 6,$$
$$13 \cdot (98 + 2) = 13 \cdot 98 + 13 \cdot 2,$$

or in general

6. For any whole numbers a, b, and c, $a \cdot (b + c) = a \cdot b + a \cdot c$.
   (Distributive Property of Multiplication over Addition)

We should also like the distributive property to apply in (Z, +, ·).

6.2 Exercises

1. For each of the following state the property for multiplication of whole numbers that justifies the equality.
   (a) $87 \times 1 = 1 \times 87$
   (b) $87 \times 1 = 87$
   (c) $(98 - 97) \times 46 = 46$
   (d) $5 \times (2 \times 83) = (5 \times 2) \times 83$
   (e) $(25 \times 38) \times 4 = (38 \times 25) \times 4$
   (f) $(38 \times 25) \times 4 = 38 \times (25 \times 4)$
2. Without computing justify:
   (a) $(43 \times 28) \times 76 = (76 \times 43) \times 28$
   (b) $87 \times (43 \times 76) = (87 \times 76) \times 43$
   (c) $8 \times (69 \times 25) = 69 \times (25 \times 8)$
3. State the commutative property for addition of

whole numbers.

4. State the associative property for addition of whole numbers.

5. What is the identity element for addition of whole numbers?

6. What is the identity element (if there is one) for each of the following systems.

(a)	$(Z_5, +)$	(f)	$(\overleftarrow{W}, +)$
(b)	$(Z_5, \cdot)$	(g)	$(Z, +)$
(c)	$(Z_6, +)$	(h)	$(W, +)$
(d)	$(Z_6, \cdot)$	(i)	$(W, \cdot)$
(e)	$(\vec{W}, +)$	(j)	$(Z, -)$

7. Compute each of the following:

(a)	$8 \times (9 \times 7)$	(d)	$(8 \times 7) \times 9$
(b)	$9 \times (8 \times 7)$	(e)	$(47 \times 73) + (47 \times 27)$
(c)	$7 \times (9 \times 8)$	(f)	$(47 \times 73) - (47 \times 27)$

*8. Using the properties of this section, prove that if r, s, t are whole numbers,

(a) $(r \cdot s) \cdot t = (r \cdot t) \cdot s$ (c) $r \cdot (s \cdot t) = (r \cdot t) \cdot s$

(b) $(r \cdot s) \cdot t = (t \cdot s) \cdot r$ (d) $r \cdot (s \cdot t) = s \cdot (t \cdot r)$

For example, exercise (a) may be done as follows:

$(r \cdot s) \cdot t = r \cdot (s \cdot t)$	Multiplication of whole num-numbers is associative.
$= r \cdot (t \cdot s)$	Multiplication of whole numbers is commutative.
$= (r \cdot t) \cdot s$	Multiplication of whole numbers is associative.

9. From your experience with multiplication of whole numbers what seems to be true if the factors are ordered and grouped differently? (The generalization referred to here is sometimes called "the rearrangement property for multiplication of whole numbers".)

10. Consider the two sets (Sandwiches and Beverages)

S = {cheese, jelly, peanut butter} which is abbreviated {c, j, p} and B = {milk, tea} which we abbreviate {m, t}.

(a) Interpret the ordered pair (j, m).

(b) List all the possible ordered pairs that can be obtained by using an element of S as the first element and an element of B as the second element of each ordered pair.

(c) How many ordered pairs did you get in (b)?

(d) List all the ordered pairs if the first element must come from B and the second from S.

(e) How many did you get now?

(f) What property seems to be illustrated here?

(g) Suppose S or B had just one element. How many ordered pairs would we now get?

(h) Suppose S or B had 0 elements. How many ordered pairs would we get? What generalization does this suggest regarding a product having 0 as a factor?

11. Compute

(a)	$7 \times (20 + 7)$	(e)	$(47 \times 39) - (47 \times 29)$
(b)	$(7 \times 20) + (7 \times 7)$	(f)	$(37 \times 43) - (27 \times 43)$
(c)	$(23 \times 87) + (23 \times 13)$	(g)	$(6\frac{1}{2} \times 8) + (6\frac{1}{2} \times 12)$
(d)	$(76 \times 38) + (24 \times 38)$	(h)	$(6\frac{1}{2} \times 3\frac{1}{2}) + (6\frac{1}{2} \times 6\frac{1}{2})$

*12. Using the fact that multiplication of whole numbers is commutative and distributive over addition, prove that for all whole numbers a, b, and c,

(a) $(b+c)a = ba + ca$	(Recall that $ba = b \cdot a$, $ca = c \cdot a$, etc.)
(b) $a(b - c) = ab - ac$	whenever b is not less than c
(c) $(b - c)a = ba - ca$	whenever b is not less than c

13. Is it true that $5 + (2 \times 4) = (5 + 2) \times (5 + 4)$?

14. Is addition distributive over multiplication in $(W, +, \cdot)$?

6.3 Multiplication for Z

In order to define multiplication as an operation in Z, we must show how to assign to each order pair (a, b) of integers a third integer c called "the product of a and b". We will use the definition of multiplication for whole numbers and the six properties we want preserved, as guides to the rule of assignment for "·" in Z. Under these circumstances, there are three cases which must considered in making our definition:

1. Both a and b are positive.
2. Both a and b are negative.
3. a is positive and b is negative.

We also want our definition to make sense in situations where the integers have applications to real life problems.

Question: Why is it unnecessary to consider the case "a negative, b positive"?

6.4 Multiplication of Positive Integers

The systems $(W, +)$ and $(\vec{W}, +)$ are isomorphic. In fact, both of these systems are isomorphic to $(P, +)$ where P is the set of positive integers. The isomorphism may be illustrated by

$$5 \longleftrightarrow \vec{5} \longleftrightarrow {}^{+}5$$

$$4 + 5 \longleftrightarrow \vec{4} + \vec{5} \longleftrightarrow {}^{+}4 + {}^{+}5$$

We already know how to multiply whole numbers. This knowledge suggests a definition of multiplication for the positive integers.

$3 \times 4 = 12$ suggests $^+3 \times {}^+4 = {}^+12$

$11 \times 14 = 154$ suggests $^+11 \times {}^+14 = {}^+154$

$8 \times 0 = 0$ suggests $^+8 \times {}^+0 = {}^+0$

These examples imply that we should make the following definitions: The product of two positive integers is the unique positive integer whose absolute value is the product of the absolute values of the factors. For every positive integer a, $a \cdot {}^+0 = {}^+0 \cdot a = {}^+0$.

6.5 Multiplication of a Positive Integer and a Negative Integer

Since (under + and ·) the positive integers behave exactly like the whole numbers, let us agree to delete the elevated plus sign. For example, instead of writing "$^+2$" we shall write simply "2" and think of 2 as being a positive integer without saying "positive". We shall write "0" rather than "$^+0$" or "$^-0$" and think of 0 as being the addition identity element for integers. Moreover, it will be more convenient to regard only the strictly positive integers as being positive and the strictly negative integers as negative. With this agreement, every integer is either positive, zero, or negative. In other words, for every integer n, exactly one of these conditions must hold

$$0 < n,\ 0 = n,\ \text{or}\ n < 0.$$

Let us now write a few computations that may suggest what the product of a positive integer and a negative integer should be.

$3 \times 3 = 9$	$3 \times 3 = 9$
$3 \times 2 = 6$	$2 \times 3 = 6$
$3 \times 1 = 3$	$1 \times 3 = 3$
$3 \times 0 = 0$	$0 \times 3 = 0$
$3 \times {}^-1 = \square$	${}^-1 \times 3 = \square$
$3 \times {}^-2 = \square$	${}^-2 \times 3 = \square$
$3 \times {}^-3 = \square$	${}^-3 \times 3 = \square$

In the left column of equalities the second factor decreased by 1 as we move down. In the right column of equalities, the first factor is being reduced by 1. In both columns the products are decreasing by 3. These lists suggest that the products for the last three lines should be $^-3$, $^-6$, and $^-9$ if the products are to continue to decrease by 3. It appears that the product of a positive integer and a negative integer should be negative regardless of which is the first of the pair. Furthermore, the absolute value of the product should again be the same as the product of the absolute values of the factors.

Therefore, if r and s are two integers, one negative and the other positive, we define the product $r \cdot s$ to be the unique negative integer with absolute value equal to $|r| \cdot |s|$. It follows from this definition that $r \cdot s = s \cdot r$.

Later we shall give other reasons for adopting this definition and suggest a mathematical basis for deriving them. Let us now see some illustrative examples.

Example 1: Compute $^-8 \times 7$

$|{}^-8 \times 7| = |{}^-8| \times |7|$

$= 8 \times 7$ (Note: Here 8 and 7 are whole numbers)

$= 56$

Since $^-8$ is negative and 7 is positive, $^-8 \times 7$ is a negative integer. Hence, $^-8 \times 7 = {}^-56$.

Example 2: Compute $9 \times {}^-6$

$|9 \times {}^-6| = |9| \times |{}^-6|$

$= 9 \times 6$

$= 54$

Therefore, $9 \times {}^-6 = {}^-54$.

Example 3: Compute $(4 \times {}^-3) \times 2$

$(4 \times {}^-3) \times 2 = {}^-12 \times 2$

$= {}^-24.$

6.6 The Product of Two Negative Integers

The only remaining products to be considered are those involving two negative integers. Once again let us try to obtain a clue by recognizing a pattern.

${}^-3 \times 3 = {}^-9$	$3 \times {}^-3 = {}^-9$
${}^-3 \times 2 = {}^-6$	$2 \times {}^-3 = {}^-6$
${}^-3 \times 1 = {}^-3$	$1 \times {}^-3 = {}^-3$
${}^-3 \times 0 = \square$	$0 \times {}^-3 = \square$
${}^-3 \times {}^-1 = \square$	${}^-1 \times {}^-3 = \square$
${}^-3 \times {}^-2 = \square$	${}^-2 \times {}^-3 = \square$
${}^-3 \times {}^-3 = \square$	${}^-3 \times {}^-3 = \square$

In the left column of equalities the second factor is being reduced by 1 in moving down. In the right column of equalities the first factor is being reduced by 1. In both columns, the products are increasing by 3. The above lists suggest that the last four products should be 0, 3, 6 and 9 if the products are to continue to increase by 3.

These examples suggest the following definitions: The product of a pair of negative integers is the unique positive integer which has absolute value equal to the product of the absolute values of the factors. For every negative integer a, $a \cdot 0 = 0 \cdot a = 0$. Later we shall give other reasons for adopting these definitions and suggest a mathematical derivation.

We can summarize our definition of multiplication of integers as follows:

For all integers r and s,

1. $|r \cdot s| = |r| \cdot |s|$.

2. If r and s are both positive or both negative, $r \cdot s$ is positive.

3. If r and s are such that one is positive and the other negative, $r \cdot s$ is negative.

4. $r \cdot 0 = 0 \cdot r = 0$.

With the above definition as rules for the assignment, multiplication is an operation on Z. That is, for each ordered pair (a, b) of integers there is a unique integer $\underline{c} = a \cdot b$ called "the product of a and b". Furthermore, it can be shown that the six properties of $(W, +, \cdot)$ stated in section 6.1 are also properties of $(Z, +, \cdot)$.

The general rules for multiplication of integers may be clarified by the following illustrative examples.

Example 1: Compute $^-3 \times {}^-4$

$$|^-3 \times {}^-4| = |^-3| \times |^-4|$$
$$= 3 \times 4$$
$$= 12.$$

Since $^-3$ and $^-4$ are both negative, the product is positive. Hence, $^-3 \times {}^-4 = 12$ (What kind of number is 12 here?)

Example 2: Compute $(^-7 \times {}^-2) \times {}^-3$

$$(^-7 \times {}^-2) \times {}^-3 = 14 \times {}^-3$$
$$= {}^-42.$$

Example 3: Compute $^-9 \times (6 \times {}^-4)$

$$^-9 \times (6 \times {}^-4) = {}^-9 \times {}^-24$$
$$= 216$$

6.7 Exercises

1. Compute:

(a) $^-23 \times 27$ (e) $^-5 \times (2 \times {}^-47)$

(b) $33 \times {}^-37$ (f) $(^-5 \times 2) \times {}^-47$

(c) $^-43 \times {}^-47$ (g) $(^-43 \times {}^-4) \times {}^-25$

(d) $^-57 \times {}^-53$ (h) $^-43 \times (^-4 \times {}^-25)$

2. Compute:

(a) $(^-17 \times {}^-7) + (^-17 \times {}^-3)$

(b) $(^-17) \times (^-7 + {}^-3)$

(c) $(^-38 \times 37) + (28 \times 37)$

(d) $(^-38 + 28) \times 37$

(e) $(^-83 \times {}^-67) + (^-27 \times {}^-67)$

(f) $(^-37 \times 73) + (37 + 73)$

*3. Suppose r, s, and t are integers.

(a) Give a rule for determining whether $(r \cdot s) \cdot t$ is positive or negative. What about $r \cdot (s \cdot t)$?

(b) Try to justify that $|(r \cdot s) \cdot t| = |r \cdot (s \cdot t)|$

(c) Give a general way of computing $|(r \cdot s) \cdot t|$

(d) Prove that

(1) $r \cdot (s \cdot t) = (r \cdot s) \cdot t$

(2) $r \cdot s = s \cdot r$

4. Let us return to considering an integer as a set of ordered pairs and see how multiplication may be defined. You recall:

$$^+3 = \{ (0,3), (1,4), (2,5), \ldots \}$$
$$^-2 = \{ (2,0), (3,1), (4,2), \ldots \}$$
$$^-6 = \{ (6,0), (7,1), (8,2), \ldots \}$$

Let us define a multiplication of two ordered pairs as follows:

$$(a, b) \cdot (c,d) = (ad+bc, ac+bd)$$

so that $(2,5) \cdot (5,3) = (6+25, 10+15)$
$= (31,25)$

Observe that (31,25) is in the set for $^-6$.

(a) Take two other ordered pairs, one from $^+3$ and one from $^-2$. Is their product in $^-6$? Reverse the order of the factors. Is their product the same?

(b) Take one ordered pair from $^-3$ and one from $^-2$. Is their product in $^+6$? Reverse the order of the factors. Is their product the same?

*(c) Check that $[^-4 \times (^-3 + {}^+2)]$ and $[(^-4 \times {}^-3) + (^-4 \times {}^+2)]$ are ordered pairs of the same integer by using in place of each integer one of its ordered pairs and then using the definitions of addition and multiplication of ordered pairs.

*(d) Check that $[(^-4 \times {}^-3) \times {}^+2]$ and $[^-4 \times (^-3 \times {}^+2)]$ are ordered pairs of the same integer by the method used in (c).

*(e) Assume the familiar properties of addition and multiplication for whole numbers. Use the ordered pairs of whole numbers,

$$(a,b),\ (c,d),\ (e,f)$$

to check that:

(1) Addition of integers

a. is commutative: $[(a,b)+(c,d)]$ and $[(c,d)+(a,b)]$ are ordered pairs of the same integer.

b. is associative

(2) The sum $(d,d)+(a,b)$ and (a,b) are ordered pairs of the same integer.

(3) The sum $(a,b)+(b,a)$ is an ordered pair of the integer 0.

(4) Multiplication of integers

a. is commutative: $(a,b)(c,d)$ and $(c,d)(a,b)$ are ordered pairs of the same integer

b. is associative

c. distributive over addition.

(5) The product $(d,d) \cdot (a,b)$ is an ordered pair of the integer 0.

(6) The product $(d,d+1) \cdot (a,b)$ and (a,b) are ordered pairs of the same integer.

5. The "least" ordered pair of an integer is the ordered pair having 0 for one of its members.

(a) Find the least ordered pair for each of the integers

(1) 7 (2) $^{-}7$ (3) 0

(b) For each of the following ordered pairs, find the integer containing it

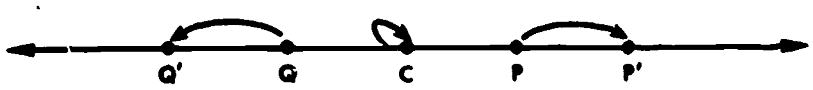
(1) (0,12) (2) (12,0) (3) (38,47)

(c) Compute each of the following, then state the least ordered pair that is in the same integer.

(1) $(0,7) + (9,0)$	(6) $(7,7) \cdot (9,0)$
(2) $(0,7) - (9,0)$	(7) $(0,1) \cdot (9,0)$
(3) $(0,7) \cdot (0,9)$	(8) $(1,0) \cdot (9,0)$
(4) $(0,7) \cdot (9,0)$	(9) $(1,0) \cdot (0,1)$
(5) $(7,7) + (9,0)$	

6.8 Dilations and Multiplication of Integers

Let us begin with a line in which one fixed point is labeled "C". Consider the following mapping of the line onto itself: The mapping assigns point C to itself, but to any other point P on the line it assigns the point P′ such that P is the midpoint of segment CP′. This mapping is illustrated by the arrow diagram



For this mapping, the distance CP′ is twice the distance CP. Thus the mapping corresponds to

$$n \longrightarrow 2n$$

which takes whole numbers into their doubles. If we denote this mapping, which doubles distances from C, by "2′" (read: 2 prime), we have

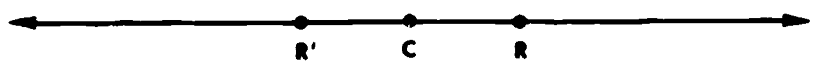
$$2': \quad P \longrightarrow P'$$
$$2': \quad C \longrightarrow C$$
$$2': \quad Q \longrightarrow Q'$$

In a similar manner we define 3′ to be the mapping that takes any point P into a point that is three times as far from C and on the same side of C as P. In general, if d is a whole number, "d′" will denote the mapping that takes any point P into a point that is d times as far from C and on the same side of C as P. Such mapping is called a dilation. Summarizing, we have:

If dilation d': $P \longrightarrow P'$, then $CP' = d \cdot CP$ and P is between C and P′, d': $C \longrightarrow C$.

Question: Does d′ map the line onto itself?

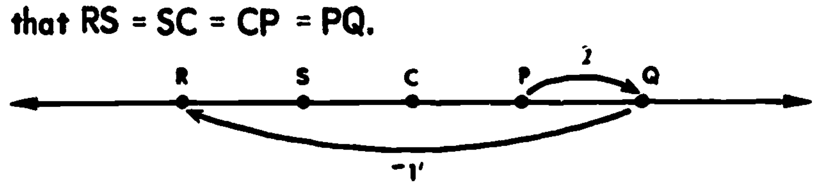
Let us now define another mapping that also leaves C fixed. This mapping takes any point R to a point on the other side of C, the same distance from C.



Since this mapping reflects R in C, it is called "the reflection in C" and is denoted "$^{-}1'$" (read: negative one prime). Such a mapping is also called a symmetry in point C because points R and R′ are located symmetrically on either side of C. However, in this chapter we shall continue to call such a mapping a reflection in a point.

$$^{-}1': \quad R \longrightarrow R'$$
$$^{-}1': \quad R' \longrightarrow R$$
$$^{-}1': \quad C \longrightarrow C$$

Let us now see what happens when we compose $^{-}1'$ with $2'$. Such maps leave C fixed, so let point P be different from Point C. Locate points Q, R, and S so that RS = SC = CP = PQ.



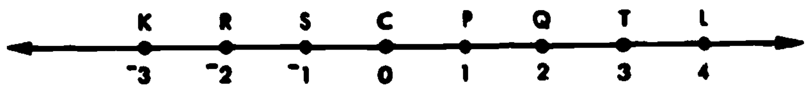
Then $2'$: $P \longrightarrow Q$ and $^{-}1'$: $Q \longrightarrow R$. The composition of $^{-}1'$ with $2'$ takes P into R via Q. Similarly, the composition of $2'$ with $^{-}1'$ takes P into R via S.

We shall see that composition of such mappings is analogous to multiplication of integers. Anticipating this analogy, let us agree to express this composition by use of the multiplication sign "x". We may now write

$$2' \times {}^{-}1': \quad P \longrightarrow R.$$
$$^{-}1' \times 2': \quad P \longrightarrow R.$$

We shall use "$^{-}2'$" as an abbreviation for "$2' \times {}^{-}1'$". Similarly, $^{-}3' = 3' \times {}^{-}1'$ and $^{-}4' = 4' \times {}^{-}1'$. We shall also say that $^{-}2'$ "contains a reflection". $^{-}3'$, $^{-}4$, $^{-}5'$, . . . also are said to contain a reflection.

Let us look at a few more examples.



Example 1: $3'$: $P \longrightarrow T$ or $1 \longrightarrow 3$

$^{-}3'$: $P \longrightarrow K$ or $1 \longrightarrow {}^{-}3$

Example 2: $3'$: $S \longrightarrow K$ or $^{-}1 \longrightarrow {}^{-}3$

$^{-}3'$: $S \longrightarrow T$ or $^{-}1 \longrightarrow 3$

Example 3: $({}^{-}2' \times 2')$: $S \longrightarrow L$ or $^{-}1 \longrightarrow 4$

because $^{-}2'$: $S \longrightarrow Q$ and

$2'$: $Q \longrightarrow L$

Note: that $^{-}4'$: $S \longrightarrow L$ or $^{-}1 \longrightarrow 4$

so that the mappings $^-2' \times 2'$ and $^-4'$ have the same effect on S.

Example 4: Let us now use only the integer names for the points.

$$(2' \cdot 3') : 1 \longrightarrow 6, \quad ^-4 \longrightarrow {}^-24$$
$$6' : 1 \longrightarrow 6, \quad ^-4 \longrightarrow {}^-24$$
$$(^-2' \times 3') : 1 \longrightarrow {}^-6, \quad ^-4 \longrightarrow 24$$
$$^-6' : 1 \longrightarrow {}^-6, \quad ^-4 \longrightarrow 24$$
$$(^-2' \times {}^-3') : 1 \longrightarrow 6, \quad ^-4 \longrightarrow {}^-24$$

What do these examples suggest?

It will be convenient to define the magnitude of such a dilation mapping. The magnitude of the mapping d′, where "d" names any integer, is the same as the absolute value of d, that is $|d|$. We shall use the same vertical bar notation to denote magnitude. Thus, $|d'| = |d|$. In particular

$$|3'| = |3| = 3$$
$$|^-3'| = |^-3| = 3$$

Let r and s be any integers; r′ and s′ their corresponding mappings. Then the composite mapping $r' \times s'$ has the following property:

$$|r' \times s'| = |r'| \cdot |s'|$$

because s′ enlarges by a factor of $|s'|$ and r′ enlarges the enlargement by a factor of $|r'|$. The net result is to enlarge by a factor of $|r'| \cdot |s'|$.

If neither r′ nor s′ contains a reflection, the composition mapping $r' \times s'$ contains no reflection. If both r′ and s′ contain reflections, then $r' \times s'$ contains no reflection. If either r′ or s′ (but not both) has a reflection, then $r' \times s'$ contains a reflection. Let us say that r′ and s′ have the same direction if either both contain reflections or neither contains a reflection. Then r′ and s′ are the same mapping if they have the same direction and magnitude.

Let us agree to call every mapping d′, where d is any integer, a dilation. The set of dilations together with the operation "x" expressing compositions determine a mathematical system which we shall denote by "(D′, ×)".

To compute the composition of two mappings will mean to express the composite mapping as a mapping without an indicated composition. Thus, the computed mapping for $^-3' \times {}^-2'$ is 6′ and we shall write $^-3' \times {}^-2' = 6'$ because $^-3' \times {}^-2'$ and 6′ have the same direction and magnitude.

The resemblance between (Z, ·) and (D′, ×) should be quite apparent by now. In the first place, there is a one-to-one correspondence between the integers Z and the dilations D′. Moreover, composition of dilations strongly suggests how we should multiply integers.

6.9 Exercises

Use the integer names for points of our number line and let our dilations be with respect to 0.

1. Into what point does 7′ map each of the following:
   (a) 6 (b) $^-6$ (c) 1 (d) 0
2. Into what point does $^-7'$ map each of the following:
   (a) 6 (b) $^-6$ (c) 1 (d) 0
3. Into what point does $2' \times {}^-3'$ map each of the following:
   (a) 1 (b) $^-1$ (c) 0 (d) 10 (e) $^-10$
4. Compute:
   (a) $^-7' \times {}^-6'$ (g) $^-35' \times {}^-35'$
   (b) $7' \times {}^-6'$ (h) $45' \times 45'$
   (c) $6' \times {}^-7'$ (i) $(2' \times {}^-3') \times 4'$
   (d) $^-6' \times 7'$ (j) $2' \times ({}^-3' \times 4')$
   (e) $^-15' \times {}^-15'$ (k) $({}^-17' \times 25') \times 4'$
   (f) $25' \times 25'$ (l) $^-17' \times (25' \times 4')$

*5. Let r′, s′, t′ be any dilations. Prove that composition of dilations
   (a) is commutative: $r' \times s' = s' \times r'$
   (b) is associative: $(r' \times s') \times t' = r' \times (s' \times t')$
   (c) has 1′ as an identity.

6. What can you say about the dilation 0′?

6.10 Another Isomorphism

We would like to have the positive integers, together with zero, behave just like the whole numbers under multiplication. In fact, it is for this reason that we are using the same symbols. "2" names either a whole number or a positive integer—we can tell which only from context. Composition of dilations that do not have reflections behaves exactly like multiplication of whole numbers. For this reason we may define the product of any pair of integers r and s by using the information we have for (D′, ×).

In order to define r · s we need to know its absolute value and its direction. Let $|r \cdot s| = |r' \times s'| = |r'| \times |s'|$ and let the direction of r · s be negative if either r′ or s′ (but not both) have a reflection; otherwise, let the direction of r · s be positive. With this definition, (Z, ·) and (D′, ×) are isomorphic.

$$r \longleftrightarrow r'$$
$$r \cdot s \longleftrightarrow r' \times s'.$$

6.11 Multiplication of Integers through Distributivity

In section 6.1 we said that it would be nice if (Z,+,·) retained the distinctive properties of (W,+,·). In order to extend the isomorphism between (W,+) and the non-negative integers, we assumed that the product of two positive integers is a positive integer. Then by observing patterns of multiplication, we were led to definitions in the cases where one or both factors are negative or zero. We found that these definitions did preserve the desired properties.

Are there other possible ways to define multiplication in Z and still retain those properties? Could such alternative definitions lead to results differing from those we have already obtained. For example, could $r \cdot 0 = r$ for every integer r ? Could the product of two negative integers turn out to be negative integer? (For instance, could $^{-}7 \cdot {}^{-}13 = {}^{-}91$?)

In this section we shall show that if "$\cdot$" is assumed to be a commutative, associative, and distributive operation, the customary rules for computing products are actually forced on us.

Let us begin by stating a basic assumption which we have been using over and over. To illustrate this assumption, which we shall soon name, consider the easy computation

$$(2 + 3) + 4 = 5 + 4$$
$$= 9.$$

The symbols "2 + 3" and "5" both name the same number so we feel free to replace "2 + 3" by "5". In the last step we replaced "5 + 4" by "9" because they both name the same number.

In mathematics we frequently replace one name for an object by another name for the same object, assuming that this kind of replacement is permitted. This assumption can be stated precisely as follows: The mathematical meaning of an expression is not changed if in this expression one name of an object is replaced by another name for the same object. This assumption will be called the Replacement Assumption or simply Replacement. We shall be making frequent use of this assumption without mentioning it.

The second assumption is that multiplication is an operation on Z. For each pair of integers r and s, there is a unique integer $r \cdot s$.

The third assumption is that multiplication of integers is commutative. For every pair of integers r and s, $rs = sr$.

The fourth assumption is that multiplication of integers is associative. For every triple of integers r, s, and t, $r(st) = (rs)t$.

The fifth assumption is that multiplication is distributive over addition. For all integers r, s, and t, $r \cdot (s + t) = rs + rt$.

The sixth and final assumption is that the product of a pair of positive integers is a positive integer and for every pair of integers r and s, the absolute value of $r \cdot s$ is equal to the product of the absolute values of r and s.

These six assumptions may be summarized as follows:

A.1 Replacement.
A.2 Multiplication is an operation.
A.3 Commutativity.
A.4 Associativity.
A.5 Distributivity.
A.6 The product of two positive integers is a positive integer.

We are now ready to prove some further useful properties of the system $(Z, +, \cdot)$:

T1: Cancellation For Addition

We will prove that for integers, whenever $x + b = y + b$ it follows that $x = y$.

If $x + b = y + b$ then it follows that $(x + b) + {}^{-}b = (y + b) + {}^{-}b$ because we are adding ^{-}b to the same number, only using different names $x + b$ and $y + b$ for this number. Using the associative property for addition of integers and Replacement, this equality may be written as

$$x + (b + {}^{-}b) = y + (b + {}^{-}b)$$

But $b + {}^{-}b = 0$. So, because of Replacement, we now write

$$x + 0 = y + 0.$$

0 is the additive identity for integers, so $x + 0 = x$ and $y + 0 = y$. Again using Replacement, we obtain $x = y$. We have thus shown that if $x + b = y + b$, then $x = y$. It also follows readily that if $b + x = b + y$, then $x = y$. We shall refer to these generalizations as Cancellation for Addition.

T2: Each Integer Has Exactly One Additive Inverse

Suppose the integer r had two inverses, say s and t. Then by definition of additive inverse,

$$s + r = 0 \text{ and } t + r = 0.$$

But if $s + r$ and $t + r$ are both 0, we have $s + r = t + r$. From cancellation for addition we have

$$s = t.$$

Hence, we can have but one additive inverse of any integer r. We shall usually denote it by "^{-}r".

We are now ready to give convincing arguments for certain rules for computing products of integers. We shall begin with a rule, which we previously adopted as a definition.

T3: For every integer r, $r \cdot 0 = 0 \cdot r = 0$

We already know that $r = r + 0$.

$r \cdot r = r \cdot r$	Multiplication is an operation
$r \cdot (r + 0) = r \cdot r$	Replacement of "r" by "r + 0"
$(r \cdot r) + (r \cdot 0) = r \cdot r$	Distributivity
$(r \cdot r) + (r \cdot 0) = (r \cdot r) + 0$	0 is the additive identity
$r \cdot 0 = 0$	Cancellation for Addition
$0 \cdot r = r \cdot 0$	Commutativity
$0 \cdot r = 0$	Replacement

This generalization or theorem says that the product of two integers is zero whenever one of the integers (or both) is zero. You recall that in Section 6.6 we defined $r \cdot 0 = 0 \cdot r = 0$. T3 shows that if we make the desired assumptions about "$\cdot$", there is really no choice in the definition of $r \cdot 0$: It must be zero! Those desired assumptions place further restrictions on the rules for computing products. These restrictions are demonstrated in T4 and T5.

T4: For every pair of integers r and s, $^-(r \cdot s) = (r \cdot {}^-s)$ and $^-(r \cdot s) = ({}^-r \cdot s)$.

This says that the additive inverse of the product of a pair of integers is the product of either one of the integers and the additive inverse of the other.

For example,

$$^-(2 \cdot 3) = ({}^-2 \cdot 3)$$

and $$^-(2 \cdot 3) = (2 \cdot {}^-3).$$

By our assumption on products of positive integers, $2 \cdot 3 = 6$. Therefore, this rule tells us that $^-2 \cdot 3 = 2 \cdot {}^-3 = {}^-6$.

A proof for rule T4 will now be given: We already know that

$$r(s + {}^-s) = (rs) + (r \cdot {}^-s)$$

because of distributivity. But $s + {}^-s = 0$, so the left number becomes $r \cdot 0$ which we know from T3 is 0. Therefore,

$$0 = (rs) + (r \cdot {}^-s)$$

Whenever a sum of two integers is 0, the integers are additive inverses of each other. Hence,

$$^-(r \cdot s) = (r \cdot {}^-s) \quad \text{(and also } (rs) = {}^-(r \cdot {}^-s)\text{)}.$$

This completes the proof that the additive inverse of a product of two integers is the product of either integer and the additive inverse of the other.

From this generalization we can conclude further that

$$^-(r \cdot {}^-s) = ({}^-r \cdot {}^-s) \qquad (1)$$

and $$^-({}^-(r \cdot s)) = ({}^-r \cdot {}^-s). \qquad (2)$$

On the left side of this last equation we have the additive inverse of the additive inverse of $(r \cdot s)$. The additive inverse of the additive inverse of an integer is the integer itself. We can see this from the following equation. If

$$t + {}^-t = 0,$$

then t and ^-t are additive inverses of the other, so that $t = {}^-({}^-t)$. Making use of this fact and (2) above, we obtain

$$r \cdot s = {}^-r \cdot {}^-s.$$

This proves the following important generalization.

T5: The product of a pair of integers is the same as the product of their additive inverses.

If <u>r</u> and <u>s</u> are negative integers, it follows that their product is the same as the product of their inverses, or a positive integer. For example,

$$^-7 \cdot {}^-4 = 7 \cdot 4 = 28.$$

6.12 Exercises

*1. Using the methods of section 6.11 prove the following:

(a) If $x + 3 = y + 3$ then $x = y$ for integers

(b) 3 has but one additive inverse.

(c) $3 \cdot 0 = 0$

(d) $3 \cdot {}^-2 = {}^-(3 \cdot 2)$

(e) $({}^-3 \cdot {}^-2) = 6$

(f) $({}^-1) \cdot r = {}^-r$ for every integer r.

2. Construct a flow chart for computing the product of a pair of integers <u>r</u>, <u>s</u>.

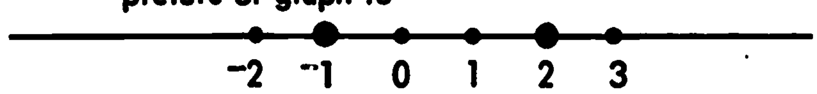
3. Compute each of the following if $r = {}^-4$, $s = {}^-7$, $t = 9$.

(a) $r + s$	(g) $(rs)\ t$	(m) $r - t$
(b) $r + {}^-s$	(h) $(rt)\ s$	(n) $^-(r - t)$
(c) $r - s$	(i) $(st)\ r$	(o) $^-r + t$
(d) $r\ (s+t)$	(j) r^2	(p) $^-2r + {}^-3t$
(e) $r\ (s - t)$	(k) r^3	(q) $r^2 + s^2$
(f) $(rs) - (rt)$	(l) $r^2\ s$	(r) $r^2 - s^2$

4. Find the solution set from the set of integers for each of the following conditions.

(a) $x^2 = 4$	(f) $(x + 2)^2 = 9$
(b) $\lvert x \rvert = 2$	(g) $(y - 3)^2 = 9$
(c) $y^2 = {}^-4$	(h) $(x + 2)\ (x - 3) = 0$
(d) $x^2 < 4$	(i) $(x + 2)^2 < 5$
(e) $\lvert x \rvert < 2$	(j) $(y - 2)^2 < 5$
	(k) $x^2 + 3x = 0$

5. Picture the solution set for each of the exercises in 4 by using a number line and enlarging dots. Thus, if your solution set is $\{{}^-1, 2\}$ its picture or graph is



*6. Prove that if the product of a pair of integers is zero, at least one of the factors is zero. (Hint: Consider the possible directions for the pair).

*7. Prove that if $r + r = 0$, then $r = 0$.

*8. Prove that: (a) If $t \neq 0$ and $rt = st$, then $r = s$. (Cancellation for Multiplication)

(b) For all integers <u>r</u>, <u>s</u>, and <u>t</u>,

(1) $r(s - t) = rs - rt$.

(2) $(s + t)r = sr + tr$.

(3) $(s - t)r = sr - tr$.

*9. Compute: (r, s, t are integers)

(a) $(r+1)(r+1)$ (f) $(r-1)(r-1)$ (k) $(r+1)(r-1)$

(b) $(s+2)(s+2)$ (g) $(s-2)(s-2)$ (l) $(s+2)(s-2)$

(c) $(t+3)(t+3)$ (h) $(t-3)(t-3)$ (m) $(t+3)(t-3)$

(d) $(r+1)(r+2)$ (i) $(r-1)(r-2)$ (n) $(r+1)(r-2)$

(e) $(s+2)(s+3)$ (j) $(s-2)(s-3)$ (o) $(s+3)(s-3)$

*10. If r, s, t are any integers and $r < s$, prove:

(a) $2r < 2s$

(b) $3r < 3s$

(c) $rt < st$ if $0 < t$

(d) $rt > st$ if $t < 0$

(e) $r^2 > 0$ if $r \neq 0$

(f) $r + t < s + t$

11. Write an equation for each of these sentences and then find all the integer solutions.

(a) the double of an integer is $^-12$.

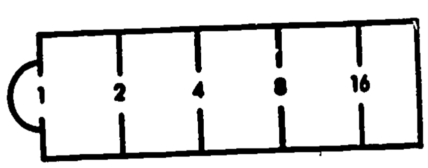
(b) the double of an integer is 3 less than the integer.

(c) the square of an integer is less than 20 but greater than 4.

(d) the sum of an integer and its successor is $^-7$.

(e) the product of an integer and its successor is 42.

12. Make two strips with scales as shown:



(a) Try to find a way of using your strips to compute products. Draw a picture showing the position of your strips for the products

(1) 2×2

(2) 2×4

(3) 2×8

(4) 4×2

(5) 4×4

(b) Notice that the scales do not show all the whole numbers. Should the exact midpoint between the markings for 2 and 4 be 3? If not, should it be more or less? Why do you think so? The strips you have constructed make a crude slide rule for multiplication.

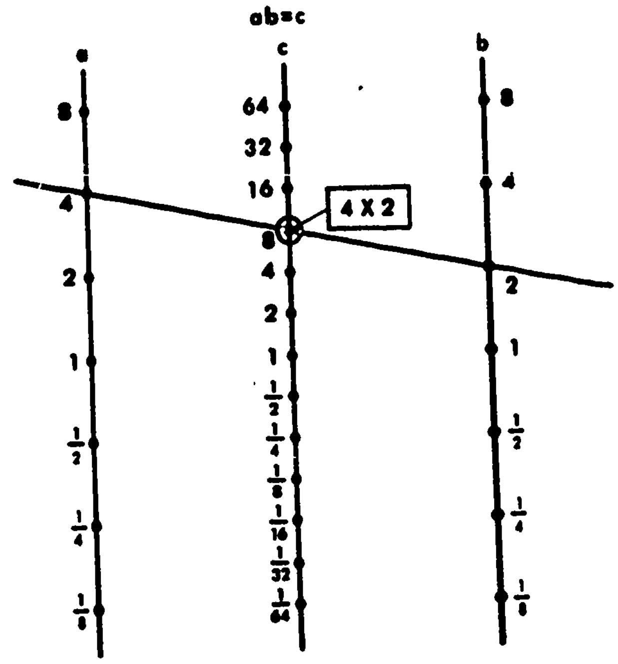
13. The figure shown is a nomogram for multiplication. The figure shows how to compute 2×4. Draw lines to show the computation for:

(a) 2×2 (e) 4×4 (i) $\frac{1}{8} \times \frac{1}{2}$

(b) 2×4 (f) 4×8 (j) $8 \times \frac{1}{6}$

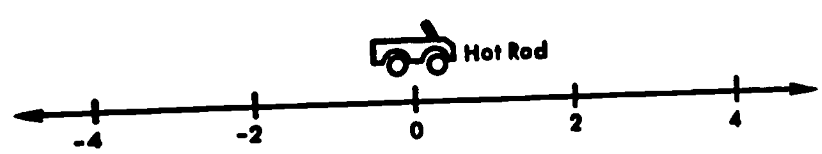
(c) 2×8 (g) 8×8 (k) $\frac{1}{4} \times \frac{1}{4}$

(d) 2×16 (h) $\frac{1}{2} \times 8$ (l) $\frac{1}{8} \times \frac{1}{8}$



14. (a) If a hot rod moves at a fixed speed of 4 feet per second to the right, what interpretation would you give to a speed of $^-4$ feet per second?

(b) If the hot rod starts at 0 on the number line (measured in units of 1 foot) and has a speed of 4 feet per second (fps), where will the hot rod be in 3 seconds? If we think of the place on the number line the hot rod is at the moment, how might we interpret the instant $^-3$ seconds?



(c) Let us agree to interpret 4×3 as a product that locates the hot rod on the number line if it starts at 0, where 4 is the speed in fps, and 3 is the number of seconds from the time it was at 0. Interpret the following products and see whether your interpretations is consistent with our rules for multiplying integers:

(1) $4 \times {}^-2$

(2) $^-4 \times 2$

(3) $^-4 \times {}^-2$

15. We have already noted the following:

$$7 + {}^-2 = 7 - 2$$

$$7 - {}^-2 = 7 + 2$$

and in general, that for any integers r and s

$r + {}^{-}s = r - s$

$r - {}^{-}s = r + s$

We shall now observe a convention that is followed universally, in view of the last two equalities. This will remove the need for using the elevated minus sign. We will use "$-s$" to mean ${}^{-}s$, and read it as "additive inverse of s". When we write "$r-s$" we may mean either the difference of r and s or the sum of r and $-s$ because either interpretation gives the same result. Assuming that we are dealing with integers, compute:

(a) $7 - 5$ (f) $(-41) \times (-39)$

(b) $5 - 7$ (g) $-3 + (-2 \times 3)$

(c) $-5 - 7$ (h) $(-7 \times 6) + (-7 \times -5)$

(d) $21 \times (-19)$ (i) $(-7 \times 6) + (-7 \times -5)$

(e) $-31 \times (29)$

16. In $(Z_7, \cdot)$ find ${}^{-}2$, ${}^{-}3$, ${}^{-}6$. Show that:

(a) ${}^{-}2 \cdot 3 = {}^{-}6$ (b) $2 \cdot {}^{-}3 = {}^{-}6$

(c) ${}^{-}2 \cdot {}^{-}3 = 6$

*17. Using the distributive property of multiplication over addition, prove: (x is an integer)

(a) $2x + 3x = 5x$ (d) $2x - 5x = -3x$

(b) $2x + x = 3x$ (e) $5x - x = 4x$

(c) $5x - 2x = 3x$ (f) $x - 5x = -4x$

18. Solve for x from among the integers:

(a) $2x+3x=20$ (f) $2x=3x+20$

(b) $2x+3x=-20$ (g) $20=2x-3x$

(c) $3x=20+2x$ (h) $|2x+x| < 7$

(d) $3x=20-2x$ (i) $|2x+3|+|x-1| < 10$

(e) $3x=2x-20$

6.13 Summary

1. In this chapter we developed and studied multiplication of integers from various points of view. The definition of the multiplication operation was motivated by the desire to extend the isomorphism between (W, +) and (Z, +), to maintain patterns previously known to hold for multiplication in W, and to preserve certain nice properties of $(W, +, \cdot)$. Multiplication received further interpretation as a composition of dilation mappings. Another development (in the exercises) used ordered pairs to define the product of integers.

2. For integers r and s, the product $r \cdot s$ is

(1) 0 if r or s is zero:

(2) positive if both r and s are positive or both r and s are negative:

(3) negative if one is positive and the other negative.

Furthermore, the absolute value of $r \cdot s$ is $|r| \cdot |s|$.

3. In section 6.11 we showed that if multiplication is assumed to be commutative, associative, and distributive, then the following must be properties of $(Z, +, \cdot)$:

T1: Cancellation for addition.

T2: Each integer has exactly one additive inverse

T3: The product of a pair of integers is 0 whenever one of the factors is 0. ($r \cdot 0 = 0 \cdot r = 0$ for all integers r)

T4: The additive inverse of the product of two integers is the product of either integer and the additive inverse of the other. (For all integers r, s, $-(r \cdot s) = (-r \cdot s)$, $-(r \cdot s) = (r \cdot -s)$.

T5: The product of a pair of integers is the same as the product of their additive inverses.

In proving these theorems, we used the Replacement Assumption and the assumption that the product of two positive integers is a positive integer.

6.14 Review Exercises

1. Compute:

(a) $9 + {}^{-}7$ (g) $|{}^{-}23 \cdot 9|$

(b) $9 - {}^{-}7$ (h) $|{}^{-}23| \cdot |9|$

(c) ${}^{-}9 - {}^{-}7$ (i) $-47 \cdot (17-25)$

(d) $({}^{-}9) \cdot ({}^{-}7)$ (j) $(39 \times {}^{-}27) - (39 \times {}^{-}17)$

(e) $9 \cdot ({}^{-}7)$ (k) $(29 \times {}^{-}7) + (29 \times {}^{-}13)$

(f) $({}^{-}12)^2$ (l) $47^2 - 48^2$

2. Find the solution set from the set of integers.

(a) $x^2 = 9$ (f) $x(x+2) = 0$

(b) $y^2 - 1 = 0$ (g) $n(n+1) = 55$

(c) $({}^{-}2)x = 8$ (h) $(x+1)^2 = 4$

(d) $r^2 < 5$ (i) $|r^3| < 10$

(e) $x^2 = {}^{-}1$ (j) $s^2 = {}^{-}s$

3. Picture on a number line the solution set for each exercise in 2.

4. Answer TRUE (T) or FALSE (F)

(a) Multiplication of integers is both commutative and associative.

(b) Multiplication of integers distributes over both addition and subtraction.

(c) Multiplication of integers by $^{-}2$ always gives a smaller integer.

(d) Subtraction of integers is associative.

(e) If a product of integers is 0, one of its factors must be 0.

(f) If a product of integers is negative, then at least one of the factors must be negative.

(g) If r, s, t are integers and $(rs)t < 0$ then r or s or t must be positive.

(h) $(Z, \cdot)$ is isomorphic to $(W, \cdot)$.

(i) $(Z, \cdot)$ is isomorphic to $(D', \times)$.

(j) If one of the factors of a product of integers is 0 then the product is 0.

(k) In $(Z_6, \cdot)$ if a product is 0 then one of the factors must be 0.

(l) $(^{-}r \cdot s) = (r \cdot {^{-}s})$

(m) $^{-}r \div s = s - r$

(n) $^{-}r + s = r - s$

(o) $^{-}(r + s) = {^{-}r} - s$

CHAPTER 7
LATTICE POINTS IN THE PLANE AND MAPPINGS ON Z X Z

7.1 Points and Ordered Pairs

In a certain North American city the city planners wanted to devise a numbering system for naming streets and avenues. They wanted a system that would make it easy for a visitor to find his way to places in the city and also a system that would allow them to add new streets and avenues without changing the system.

The avenues were to be parallel to one another and so were the streets. Each intersection then could (hopefully) be named by an ordered pair of natural numbers by agreeing that the avenue should be named first.

The planners selected an avenue and street that intersected in the city center. They named the avenue, "100 avenue", and the street, "100 street". In this way they felt that there would be sufficient street and avenue numbers on either side of the center of town to accommodate growth.

Question: What problem might arise with the system of naming avenues and streets, if the city should expand more than 99 blocks from the city center in the direction of decreasing street numbers or decreasing avenue numbers?

If the city planners could have persuaded the residents to accept integers as names for avenues and streets, the problem would be solved, for the set of integers has no least or greatest member. No matter how many new streets might be built at either end of the city there would always be enough names without changing the system. Figure 7.1 illustrates such a system:

The point labeled "(0,0)" in Figure 7.1 represents the intersection of zero avenue and zero street. This is "city center." We can name each intersection represented in Figure 7.1 with an ordered pair of integers. The first component of the pair names the avenue and the second the street. For example, intersection A is named (–2,2).

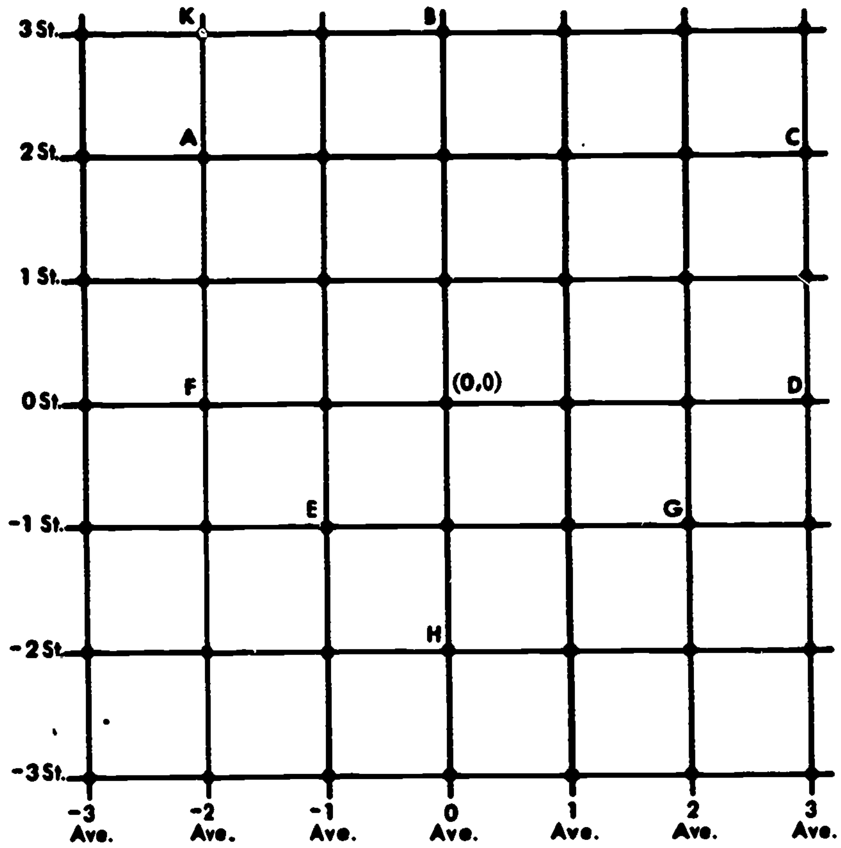


Figure 7.1

7.2 Exercises

1. Give the ordered pair of integers for each of the following intersections represented in Figure 7.1. In each case list the avenue first and then the street:

(a) K	(b) H	(c) G	(d) F
(e) E	(f) B	(g) C	(h) D

2. In traveling from intersection F to intersection C by the shortest route through D:
   (a) How many blocks would you travel?
   (b) Name the ordered pair for each intersection on the route.
   (c) How could you use the fact that F is on –2 avenue and D is on 3 avenue to find the distance in blocks from F to D, considering that they are on the same street?
   (d) Since D and C are on the same avenue, how can you use the fact that D is on 0 street and C is on 2 street to find the distance D to C in blocks?

3. In general:
   (a) How can you find the distance between two intersections on the same street?

(b) How can you find the distance between two intersections on the same avenue?

7.3 Some Important Properties of Points, Lines and Planes

The street plan illustrated in Figure 7.1 suggests many geometric ideas. It suggests two sets of equally spaced parallel lines so arranged that each line in one set intersects each line in the other. It also suggests a correspondence between the intersect points and ordered pairs of integers. This set of intersection points will be called a set of *lattice points*. The set of all ordered pairs of integers will be called Z X Z (read: Z cross Z).

The following properties of points and lines are useful in establishing a correspondence between Z X Z and a set of lattice points in a plane.

(1) Through two points, P and Q, ($P \not= Q$) there is exactly one line.

Figure 7.2

(2) Two lines, r and s, are called intersecting lines, if they have exactly one point in common.

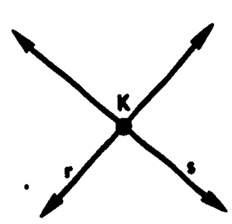


Figure 7.3

(3) Two lines are parallel if they are the same line or if they are in the same plane and have no points in common.

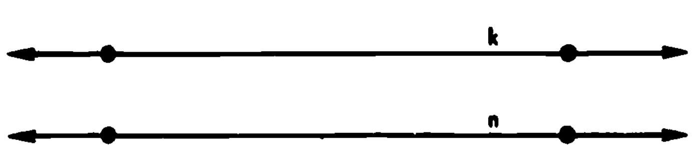


Figure 7.4

(4) Two lines in the same plane are either parallel or they are intersecting lines.

(5) Through a point P, there is exactly one line parallel to a given line m.

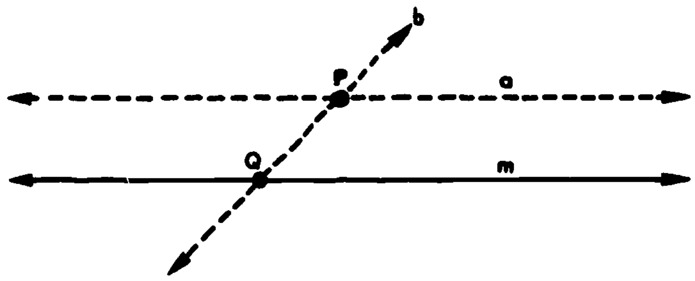
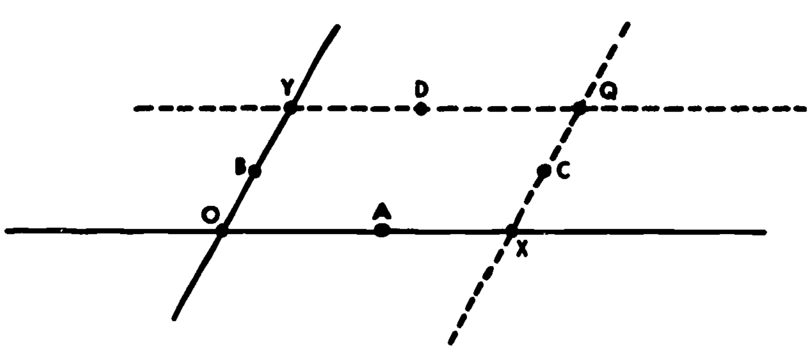


Figure 7.5

These properties are quite simple, but they are basic geometric properties which will be applied many times. We apply them first to the following problems.

In Figure 7.6, $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ are intersecting lines. X is a point on $\overleftrightarrow{OA}$ (not the same as O) and Y is a point on $\overleftrightarrow{OB}$ (not the same as O). Given these conditions on O, X, and Y, is there a unique point Q, so that OYQX is a parallelogram? If the answer is yes, then we have defined an operation on pairs of points on $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ respectively. If we designate the operation by *, then X*Y = Q.

Figure 7.6



7.4 Exercises

1. Use the picture in Figure 7.6 as a reference while answering the following questions. The answers to these questions may help you to justify the fact that * is an operation.

   (a) Suppose that $\overleftrightarrow{XC}$ is parallel to $\overleftrightarrow{OB}$ and $\overleftrightarrow{YD}$ is parallel to $\overleftrightarrow{OA}$. What kind of figure is OYQX?

   (b) Are there any other lines through X which are parallel to $\overleftrightarrow{OB}$? Are there any other lines through Y which are parallel to $\overleftrightarrow{OA}$?

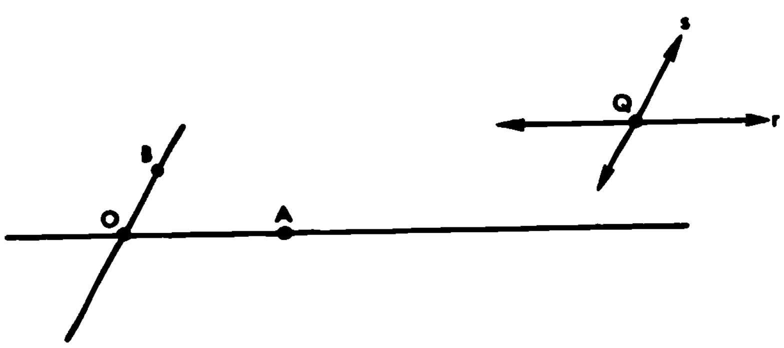
   (c) What is the reason that $\overleftrightarrow{XC}$ and $\overleftrightarrow{YD}$ must intersect. In other words, why can't $\overleftrightarrow{XC}$ and $\overleftrightarrow{YD}$ be parallel?

(d) If $\overleftrightarrow{XC}$ and $\overleftrightarrow{YD}$ intersect in Q, do they intersect in any other point?

2. In Exercise 1 you answered questions that show that, for the pair of points X, Y (as specified), there is one and only one Q such that $X*Y = Q$. Now the question is, "Given a point Q (not on $\overleftrightarrow{OA}$ or $\overleftrightarrow{OB}$) is there one and only one pair of points X, Y such that $X*Y = Q$?"

   Refer to Figure 7.7 and the properties in 7.3 in answering the following questions related to the one above.

Figure 7.7



(a) Suppose that r contains Q and is parallel to $\overleftrightarrow{OA}$. Is r the only line through Q which is parallel to $\overleftrightarrow{OA}$? What property justifies your answer?

(b) Suppose that s contains Q and is parallel to $\overleftrightarrow{OB}$. Is s the only parallel to $\overleftrightarrow{OB}$ which contains Q? Justify your answer.

(c) Does r intersect $\overleftrightarrow{OB}$? If r did not intersect $\overleftrightarrow{OB}$, what two lines through O would both be parallel to r? What property would this contradict?

(d) Does s intersect $\overleftrightarrow{OA}$? Justify your answer.

(e) Can r intersect $\overleftrightarrow{OB}$ in more than one point?

(f) Can s intersect $\overleftrightarrow{OA}$ in more than one point?

(g) What conclusion can you draw from your answers to the preceding questions about X,Y,Q and the operation * ?

3. Use properties 1,2, and 3 of section 7.3 to give an argument in favor of property 4.

4. In Figure 7.3 what is the intersection point of r and s? Suppose r and s had a second point of intersection. How can you use property 1 to show that r and s are then the same line?

5. In Figure 7.5, a is a line which contains P and is parallel to m. If b is another line which contains P and b is not the same line as a, what property can you use to show that b is not parallel to m?

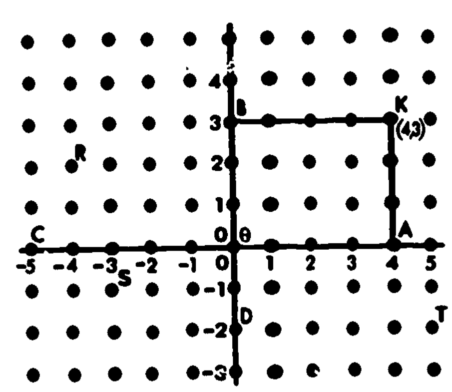
7.5 Assignment of Ordered Pairs of Integers to Lattice Points

In Figure 7.1 you have an example of an assignment of ordered pairs of integers to intersections of avenues and streets. This physical example suggests many geometric ideas and might help to keep your thinking straight about the mechanics of assigning numbers to points. In this section we will use the parallelogram operation, * , and the idea of parallel projection to develop ideas about the correspondence between Z X Z and an infinite set of lattice points.

Refer to Figure 7.8 as we illustrate and describe the assignment of an ordered pair of integers to one particular lattice point. Notice that integers have been assigned to equally spaced points on $\overleftrightarrow{\Theta A}$ and on $\overleftrightarrow{\Theta B}$ with zero assigned to Θ in both assignments. We will refer to these points by letter names, by the assigned integers, or simply as points.

Imagine a set of lines each of which is parallel to $\overleftrightarrow{\Theta B}$ and intersects $\overleftrightarrow{\Theta A}$ in a point which is assigned an integer. Also imagine a set of lines each of which is parallel to $\overleftrightarrow{\Theta A}$ and intersect $\overleftrightarrow{\Theta B}$ in a point which is assigned an integer. Since $\overleftrightarrow{\Theta A}$ and $\overleftrightarrow{\Theta B}$ intersect, each line in the set parallel to $\overleftrightarrow{\Theta A}$ intersects each line in the set parallel to $\overleftrightarrow{\Theta B}$. These intersections are the lattice points to which we will assign ordered pairs of integers.

Figure 7.8



You will see in Figure 7.8 that we have assigned the ordered pair (4,3) to point K. We will call 4 the *first coordinate* of K and 3 the *second coordinate* of K. $\overleftrightarrow{BK}$ is parallel to $\overleftrightarrow{\Theta A}$ and $\overleftrightarrow{KA}$ is parallel to $\overleftrightarrow{\Theta B}$. What kind of a geometric figure is ΘBKA?

Note that $A*B = K$. What integer has been assigned to A? to B? How can you use the operation to assign coordinates to lattice points in Figure 8 if the points are not on $\overleftrightarrow{\Theta A}$ or $\overleftrightarrow{\Theta B}$? What are the coordinates or R? S? T?

Point K obtained its first coordinate from $\overleftrightarrow{\Theta A}$ and its second coordinate from $\overleftrightarrow{\Theta B}$. We will call $\overleftrightarrow{\Theta A}$ the first axis, or more simply, the *x-axis* in our *coordinate system* and $\overleftrightarrow{\Theta B}$ the second axis or *y-axis*. "(x,y)" will be used to represent the coordinates of any point in the set of lattice points, or the point itself.

You can think of the lattice points as being arranged in "rows" and "columns", the "rows" and "columns" being sets of colinear points, i.e. points in the same line. Assign the same first coordinate to each point in the same "column", and assign the same second coordinate to each point in the same "row". Then when you assign an integer to a lattice point, X, on the x-axis, you assign it as the first coordinate of every lattice point in the column containing X. Also when you assign an integer to a lattice point, Y, on the y-axis you assign it as the second coordinate of every point in the row containing Y.

Question: What is the first coordinate of every lattice point on $\overleftrightarrow{KA}$?

What is the second coordinate of every lattice point on $\overleftrightarrow{BK}$?

The method of assigning the same first coordinate to each lattice point in a column and the same second coordinate to each lattice point in a row is illustrated in Figure 7.9.

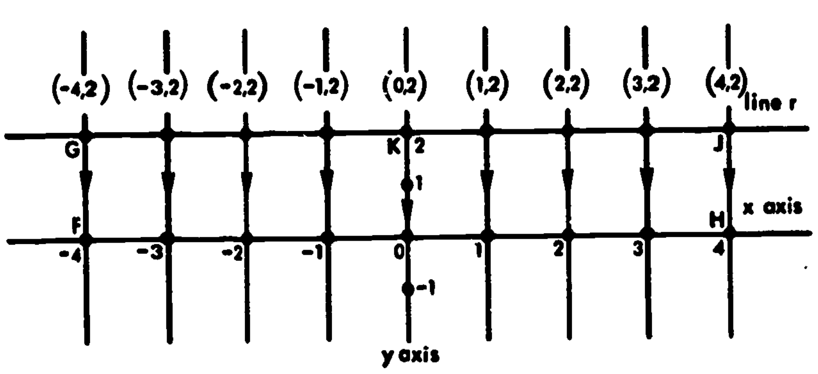


Figure 7.9

In Figure 7.9 the lattice points on line r have been mapped onto the integers assigned to lattice points on the x-axis. This is an illustration of *parallel projection* of a set of points on a first line onto a set of points on a second line which is parallel to the first. The idea of parallel projection plays a role in each of the methods that we used to assign coordinates to lattice points. Note that point K is the intersection of line r with the y-axis and that point K was assigned the integer 2.

Question: What is the second coordinate of each lattice point on line r?

What is the first coordinate of each point on $\overleftrightarrow{FG}$?

What is the first coordinate of each point on $\overleftrightarrow{HJ}$?

Figure 7.10 illustrates some of the terminology that we will use in the following sections in connection with coordinates.

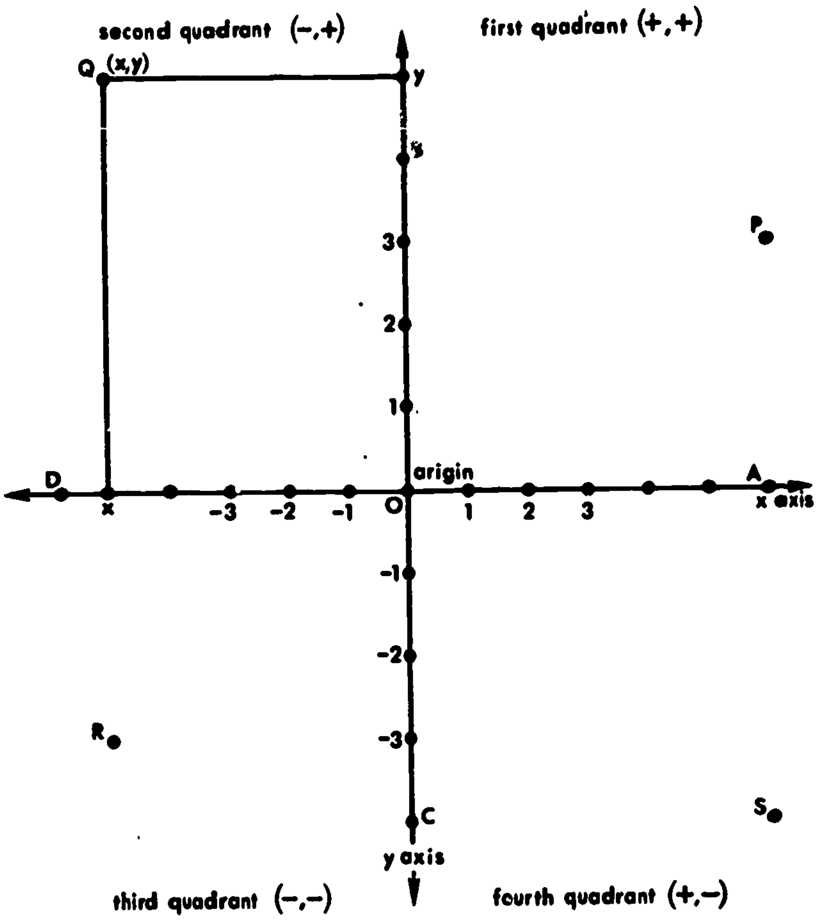


Figure 7.10

Figure 7.11, represents a *ray*. The ray includes E, the *endpoint*, and all the points of line $\overleftrightarrow{EF}$ on the "F side" of E. (represented by the solid part of the sketch)

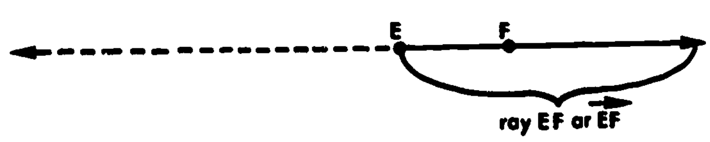


Figure 7.11

7.6 Exercises

1. (a) Locate $\overrightarrow{OA}$ in Figure 7.10. Zero has been assigned to point O. How can you describe the integers that have been assigned to the other lattice points on $\overrightarrow{OA}$?

   Note: $\overrightarrow{OA}$ is called, "the positive x-axis."

   (b) What would you call $\overrightarrow{OD}$? $\overrightarrow{OB}$? $\overrightarrow{OC}$?

   (c) Locate the part of Figure 7.10 labeled "first quadrant." The points in the first quadrant have coordinates that are both positive. For point P in the first quadrant, $x > 0$ and $y > 0$. What can you say about point Q in this respect? point R? point S?

2. We can say that $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ are the boundaries of the first quadrant. What are the boundaries of the second quadrant? the third quadrant? the fourth quadrant?

Figure 7.12 illustrates an important fact about coordinates of points on a line parallel to one of the axes.

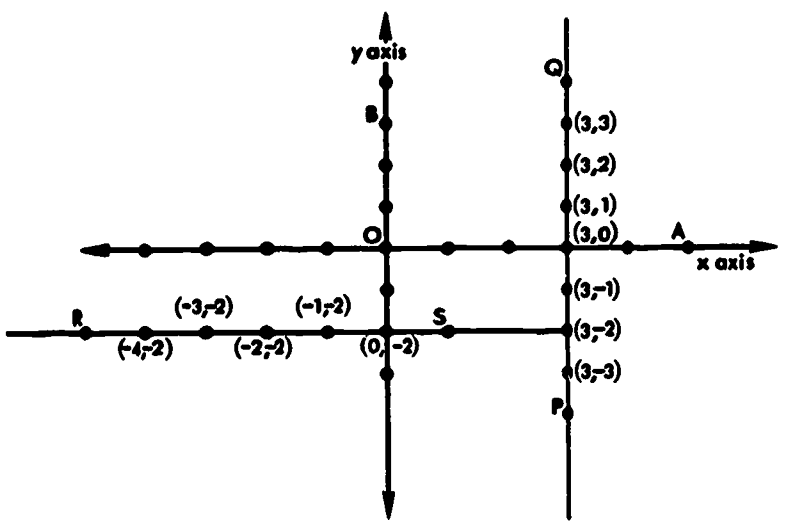


Figure 7.12

3. $\overleftrightarrow{PQ}$ is parallel to the y-axis. What do you notice about the first coordinates of points on $\overleftrightarrow{PQ}$?

4. $\overleftrightarrow{RS}$ is parallel to the x-axis. What do you notice about the second coordinates of points on the line $\overleftrightarrow{RS}$?

5. What two generalizations can you make about lines parallel to one of the axes?

6. What part of our procedure for assigning coordinates built this property into our system?

7.7 Summary of Assignment of Coordinates

The purpose of the sections on assignment of coordinates is:

(1) To give you some idea of the association of pairs of integers with lattice points:

(2) To give you some idea of the geometric properties upon which this association depends:

(3) To give you a working knowledge of the use of coordinates to aid you in the subsequent study of both algebra and geometry.

A summary of some of the main features of the assignment of ordered pairs of integers to lattice points follows:

(1) We start with a pair of intersecting lines, $\overleftrightarrow{AO}$ and $\overleftrightarrow{OB}$, called axes;

(2) We assume that there is a 1 – 1 correspondence between the set of integers and a set of equally spaced points on each axis, with zero assigned to the intersection O;

(3) Through each point on $\overleftrightarrow{OA}$, to which an integer is assigned, there is exactly one line parallel to $\overleftrightarrow{OB}$, and through each point on $\overleftrightarrow{OB}$, to which an integer is assigned, there is exactly one line parallel to $\overleftrightarrow{OA}$;

(4) The set of lattice points is precisely the set of intersections of lines, one of which is parallel to $\overleftrightarrow{OA}$ and the other to $\overleftrightarrow{OB}$ as described in (3). Each such pair of lines intersects in exactly one lattice point;

(5) On any line parallel to $\overleftrightarrow{OB}$ (including $\overleftrightarrow{OB}$), as described above, assign to each lattice point the integer for the point in which it intersects $\overleftrightarrow{OA}$. Call this integer the first coordinate of these lattice points.

(6) On any line parallel to $\overleftrightarrow{OA}$ (including $\overleftrightarrow{OA}$), as described above, assign to each lattice point the integer for the point in which it intersects $\overleftrightarrow{OB}$. Call this integer the second coordinate of these lattice points.

(7) Because two intersecting lines intersect in exactly one point, and because through any point there is exactly one line parallel to a given line, the above assignment of pairs of integers to lattice points is a one-to-one correspondence.

7.8 Exercises

For the following exercises you will need some lattice paper (perhaps your teacher will have a supply dittoed), some colored pencils, and a ruler. The lattice paper should have at least eleven rows of dots and eleven columns of dots. Draw a line through a horizontal row of dots for the x-axis and a line through a column of dots, for example as in Figure 10, for the y-axis. Ordinary graph paper can also be used.

1. Draw a line with colored pencil through the sets of points that a satisfy the following conditions. Use a different color for each condition in a group and a different sheet of lattice paper for each group:

Group 1: (a) The first coordinate is equal to the second coordinate.

(b) The 1st coordinate is the additive inverse of the second.

Group 2: (c) The sum of the coordinates of the point is 5.

(d) The sum of the coordinates is 3.

(e) The sum of the coordinates is –3.

(f) The sum of the coordinates is –5.

Group 3: (g) The first coordinate minus the second is 2.

(h) The first coordinate minus the second is –1.

Group 4: (i) The first coordinate equals 2.

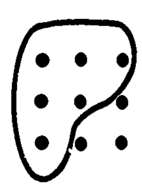
(j) The first coordinate equals –2.

(k) The second coordinate equals 4.

(l) The second coordinate equals –4.

Group 5: (m) The absolute values of the coordinates are equal.

2. For each condition listed in this exercise use a different color to draw a closed curve enclosing just those points, represented on your graph or lattice paper, that satisfy the condition, e.g.



(a) The first coordinate is less than the second.

(b) The first coordinate is greater than the second.

(c) The sum of the coordinates is greater than 5.

(d) The sum of the coordinates is less than –5.

(e) The first coordinate is less than –2.

(f) The first coordinate is greater than 3.

(g) The second coordinate is less than –4.

(h) The second coordinate is greater than 3.

7.9 Conditions on Z X Z and their Graphs

The set of ordered pairs that satisfies any one of the conditions in Exercise 1 or 2 in Section 7.3 is called the *solution set of the condition*. For example, the solution set of the condition "The sum of the coordinates is five", would include

{(0,5), (1,4), (2,3), (3,2), (4,1), (5,0), 6,–1), (7,–2) ... (–1,6), (–2,7), (–3,8), ... }

The set of lattice points associated with these ordered pairs is called the *graph* of the solution set, or sometimes the graph of the condition. The graph of the above solution set is represented in Figure 7.12 by the circled points in the lattice:

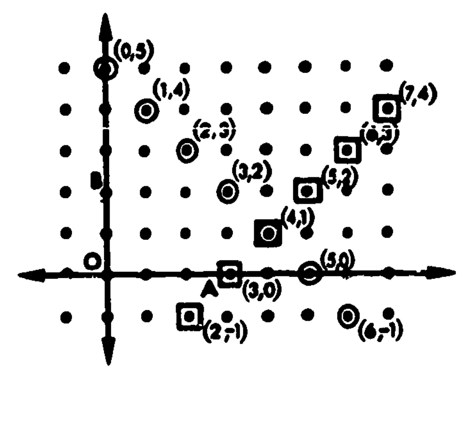


Figure 7.13

Notice that in Figure 7.12 the graph of the condition "The first coordinate is 3 more than the second coordinate" is displayed by enclosing the points in squares. (Very often it is effective to display the graphs of different conditions by using different colors to enclose the points.)

Questions: Which point is enclosed by both a circle and a square? Is 4+ 1 equal to 5? Is 4 – 1 equal to 3? Does (4,1) satisfy both conditions?

Part of our study of mathematics is learning to express mathematical ideas in the symbolism of mathematics. You have previously used "x" to express the first coordinate of a point and "y" to express the second coordinate of a point.

Therefore, instead of writing "The sum of the coordinates is 5", we can write "x + y = 5". Instead of writing "The first coordinate is 3 more than the second coordinate", we can write "x = y + 3". If we are interested in the pair of numbers that satisfies both of those conditions, we can write, "x + y = 5 *and* x = y + 3". This new condition is made up of two conditions connected by "and". The solution set is {(4,1)} and the graph is a set containing only one point. This point is called the *intersection* of the two graphs and {(4,1)} is called the intersection of the two solution sets. The sentences that we write to represent conditions are often called "open sentences".

7.10 Exercises

1. Translate the following conditions to the form used above, making use of the symbols "x", "y", "=", etc.

(a) The first coordinate is equal to the second coordinate. (Ans. x = y)

(b) The first coordinate is the additive inverse of the second coordinate. (Ans. $x = -y$)

(c) The sum of the coordinates is three.

(d) The sum of the coordinates is -3.

(e) The sum of the coordinates is -5.

(f) The difference of the first and second coordinates (in that order) is 2.

(g) The difference of the first and second coordinates is -1.

(h) The first coordinate equals 2.

(i) The first coordinate equals -2.

(j) The second coordinate equals 4.

(k) The absolute values of the coordinates are equal.

2. Draw the graphs for the open sentences you wrote in Exercise 1.

3. Translate the following back into words: (in terms of coordinates)

(a) $x + 6 = y$

(b) $y - x = 3$

(c) $y = |x|$

(d) $y = x - 2$

(e) $y = |x - 3|$

(f) $x = 7$

(g) $y = 1$

4. Using ">" for "greater than" and "<" for "less than", translate the sentences of Section 7.7 Exercise 2 into mathematical symbols.

5. Translate the following into mathematical symbols:

(a) The second coordinate is the product of 2 and the first coordinate.

(b) The first coordinate is the product of 2 and the second coordinate.

(c) The second coordinate is the product of 3 and the first coordinate.

(d) The first coordinate is the product of 3 and the second coordinate.

6. Describe the following conditions in words:

(a) $y = 5x$ (c) $y = x^2$ (e) $y < 0$ (g) $x \cdot y = 6$

(b) $x = 5y$ (d) $y = 0$ (f) $x > 0$ (h) $2x = 3y$.

7. For each of the conditions in Exercise 6, list four members of Z X Z that satisfy the condition. For example, (1,5), (· 10), (−1, −5) and (0,0) are four members of Z X Z that satisfy 6 (a).

8. Use the same piece of lattice paper to graph each of the following conditions. Use a different color for each condition to circle the points that satisfy the condition.

(a) $y = x$ (c) $x = 2y$ (e) $y = 0$

(b) $y = 2x$ (d) $x = 0$

9. What is the intersection (common point) of the graphs in Exercise 8? Which graph was included in the x-axis? the y-axis? Which of the graphs were contained in a line other than an axis?

10. Translate the following into mathematical symbols:

(a) The second coordinate is 1 more than twice the first coordinate.

(b) The first coordinate is 5 less than 3 times the second coordinate.

11. Describe the following conditions in words.

(a) $y = x + 1$ (c) $y = x + 2$

(b) $y = x - 1$ (d) $y = x - 2$

12. For each condition in Exercise 11, draw a line through the points that satisfy the condition. Use the same piece of lattice paper for all lines.

13. In what way were the four lines in Exercise 12 alike? List the coordinates of the points in which the lines intersected the y-axis. Note the similarity between these coordinates and the conditions as expressed in Exercise 11.

7.11 Intersections and Unions of Solution Sets

All of the points satisfying the condition $x > 0$ are located on the same side of the y-axis. We will designate this side of the y-axis by "A". The set of points satisfying the condition $y > 0$ are located on the same side of the x-axis. Call this set B.

Two conditions joined by a connective such as "and" is one example of a compound condition. The set of points which satisfy the *compound condition*, "$x > 0$ and $y > 0$" is the set containing these points that satisfy both "$x > 0$" and "$y > 0$". This set is the intersection of A and B. A point is in the intersection of two sets only when it is in both sets. Figure 7.14 illustrates the relationship of sets A, B and A intersection B (often written $A \cap B$).

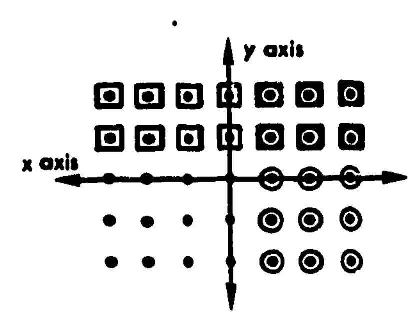


Figure 7.14

Points in A are in circles. $(x > 0)$

Points in B are in squares. $(y > 0)$

Points in A B are in circles and squares. $(x > 0$ and $y > 0)$

Let C be the set of points for "$x < 0$".

Let D be the set of points for "$y < 0$".

Illustrate C, D and $C \cap D$ in a diagram such as Figure 14.

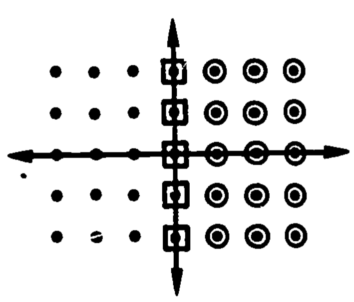
Repeat the preceding instructions for A and D: B and C.

Which quadrant contains $A \cap B$? $C \cap D$? $A \cap D$? $B \cap C$?

List the coordinates for two points in $A \cap B$: $C \cap D$: $A \cap D$: $B \cap C$.

All of the points satisfying the condition "$x = 0$" are on the y-axis. Call the set of all such points E. The solution set of the compound condition "$x > 0$ <u>or</u> $x = 0$" contains those "points" which satisfy either "$x > 0$" <u>or</u> "$x = 0$" or both. This set is the *union* of A and E, written $A \cup E$. Figure 7.15 illustrates this set relationship:

Figure 7.15



Points in A are enclosed by circles. $(x > 0)$

Points in E are enclosed by squares. $(x = 0)$

Points in $A \cup E$ are enclosed. $(x > 0$ <u>or</u> $x = 0)$

A simpler notation for "$x > 0$ or $x = 0$" is:

"$x \geq 0$" and is read,

"x is greater than or equal to zero."

7.12 Exercises

1. Describe (in terms of the axes and quadrants) the location of the points that satisfy the following conditions:

(a) $y = 0$ and $x > 0$. Ans. On the x-axis to the right of the origin.

(b) $y \geq 0$ and $x < 0$. Ans. On the x-axis to the left of the origin and in the second quadrant.

(c) $y < 0$.

(d) $y < 0$ <u>and</u> $x < 0$.

(e) $y < 0$ or $x < 0$.

(f) $x \geq 0$ <u>and</u> $x \leq 0$.

2. In this exercise try to locate the points in the graph of the compound conditions without first graphing each simple condition separately. Do all parts of this exercise on one sheet of lattice paper.

(a) $x \geq 0$ and $x = y$.

(b) $x < 0$ <u>and</u> $x = -y$.

(c) $(x \geq 0$ <u>and</u> $x = y)$ <u>or</u> $(x < 0$ and $x = -y)$

3. Follow the instructions of Exercise 2:

(a) $x \geq -1$ <u>and</u> $y = x + 1$.

(b) $x < -1$ <u>and</u> $y = -(x + 1)$

(c) $(x \geq -1$ <u>and</u> $y = x + 1)$ <u>or</u> $(x < -1$ <u>and</u> $y = -(x + 1))$

4. (a) $(x \geq 0$ <u>and</u> $y \geq 0$ <u>and</u> $x + y = 5)$

(b) $(x < 0$ <u>and</u> $y \geq 0$ <u>and</u> $y - x = 5)$

(c) $(x \geq 0$ <u>and</u> $y \geq 0$ <u>and</u> $x + y = 5)$ or $(x < 0$ <u>and</u> $y \geq 0$ <u>and</u> $y - x = 5)$

5. (a) $(x \leq 0$ <u>and</u> $y \leq 0$ <u>and</u> $x + y = -5)$

(b) $(x \geq 0$ <u>and</u> $y \leq 0$ <u>and</u> $x - y = 5)$

(c) $(x \leq 0$ <u>and</u> $y \leq 0$ <u>and</u> $x + y = -5)$ <u>or</u> $(x \geq 0$ <u>and</u> $y \leq 0$ <u>and</u> $x - y = 5)$

6. (a) $y \geq x$ <u>and</u> $y \leq x + 3$

(b) $y \leq x$ <u>and</u> $y \geq x - 3$

(c) $(y \geq x$ <u>and</u> $y \leq x + 3)$ or $(y \leq x$ <u>and</u> $y \geq x - 3)$

7.13 Absolute Value Conditions

In chapters 4 and 6 you thought of the absolute value of an integer as a whole number. This made it possible to make use of the properties of the whole numbers while developing properties of addition and multiplication of integers. You learned in those chapters that the set of whole numbers is isomorphic with respect to "+" and "·" to the set of non-negative integers. This means that they "behave" the same in addition and multiplication. In working with conditions involving absolute value in Z X Z where the solutions are ordered pairs of integers, (x,y), we will think of the absolute value of an integer as a non-negative integer as defined by the statements below:

(a) The absolute value of zero is zero.

(b) The absolute value of a positive integer is that positive integer.

(c) The absolute value of a negative integer is the additive inverse of that negative integer.

This covers all possibilities since $x = 0$, $x > 0$ or $x < 0$, if x is an integer.

A more compact definition is:

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x > 0 \end{cases}$$

For example:

If $x = 5$, $|x| = 5$ since $5 > 0$:

If $x = 0$, $|x| = 0$ since $0 = 0$:

If $x = -3$, $|x| = 3$ since $3 < 0$ and $-(-3) = 3$.

Another example:

Suppose $|x| = 3$.

From the definition $|x| = x$ or $|x| = -x$, therefore, substituting 3 for $|x|$, in the line above,

$$3 = x \text{ or } 3 = -x.$$

You see that we started with $|x| = 3$ and found as a result the compound condition "$x = 3$ or $x = -3$". The solution set of this condition is the union of the solution sets of the two simple conditions.

On a line this solution set is simply a pair of points. In the set of lattice points, however, a more interesting situation develops. In Figure 7.16 the points for which $x = 3$ or $x = -3$ are circled.

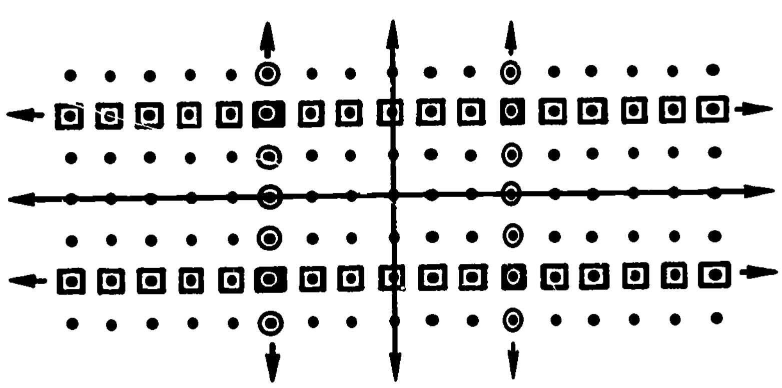


Figure 7.16

Furthermore, suppose $|y| = 2$, then $y = 2$ or $y = -2$. In Figure 16 the points for which the second coordinates are 2 or -2 are enclosed in squares. In what way is the graph of $|x| = 3$ and $|y| = 2$ indicated?

7.14 Exercises

1. What are the following?

   (a) $|-7|$ (b) $|15|$ (c) $|0|$ (d) $|-1|$ (e) $|999|$.

2. Graph the following on the same lattice.

   (a) $|x| = 4$ (c) $|x| = 4$ and $|y| = 1$.

   (b) $|y| = 1$ (d) $|x| = 4$ or $|y| = 1$.

   (e) Describe how the graphs in (c) and (d) are determined by the graphs in (a) and (b).

3. Draw the graph of $y = |x|$. Remember that if $x \geq 0$, $y = x$ and if $x < 0$, $y = -x$. $x \geq 0$ simply states that the points are to the right of the y-axis or on the y-axis. $x < 0$ states that the points are to the left of the y-axis.

4. Draw the graph of $y - |x + 1| = 0$. (i.e. $y = |x + 1|$)

   Hint:

   $$|x + 1| = \begin{cases} x + 1, \text{ if } x \geq -1 \\ -(x + 1), \text{ if } x < -1 \end{cases}$$

   Also see Exercise 3 section 7.12.

5. Graph the following:

   (a) $y = 2|x|$ (To the right of the y-axis, this becomes $y = 2x$: to the left, $y = -2x$)

   (b) $y = 3|x|$

   (c) $y = -2|x|$

6. Graph the following:

   (a) $y = |x| + 1$ (Why can you think of this as the graph of $y = |x|$ translated one space away from the x-axis?)

   (b) $y = |x| - 2$

7. Graph $|x| + |y| = 5$.

7.15 Lattice Point Games

1. The Game of Caricatures

   It is interesting to see what happens to a graph or picture when you change the angle at which the x-axis and y-axis intersect. For example see what happens to "square-head" when you change the angle of the axes:

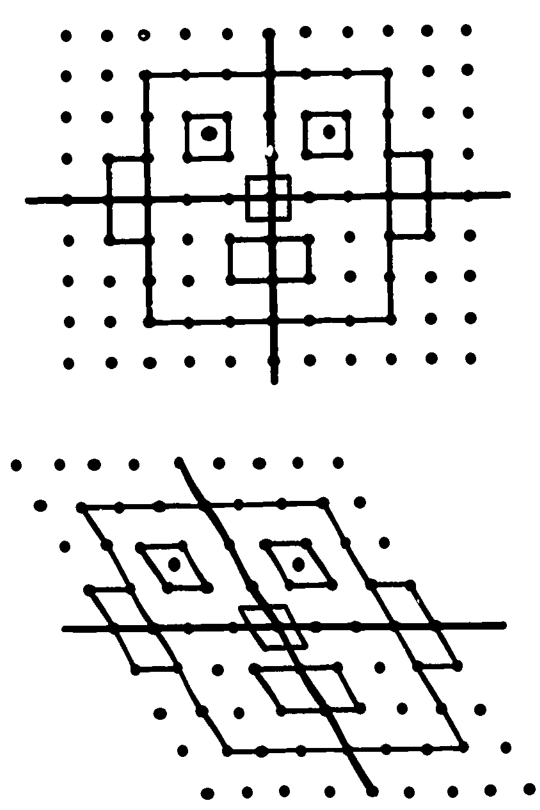


Figure 7.17

What do you think would happen to a circle if you draw it on one grid and then transfer it to another by connecting points with the same coordinates?

Transfer the "man in the moon", pictured in Figure 7.18, to another grid with the axes at a considerably different angle, e.g. X. Use the coordinates of points on the picture to make the transfer.

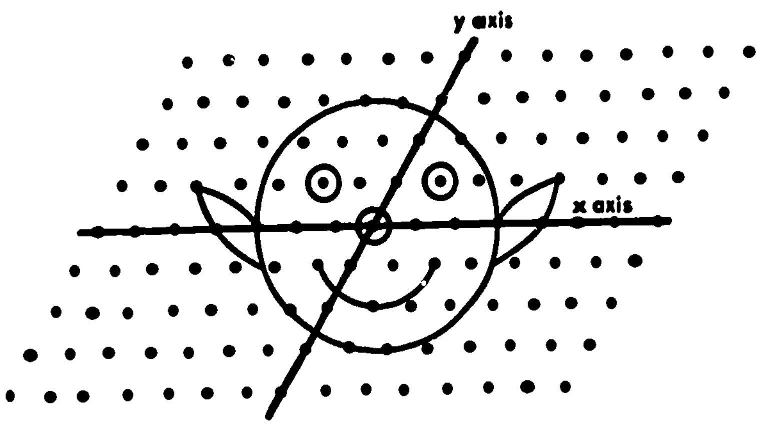


Figure 7.18

Remember that when you find the second coordinate you have to count the points along a "slanted" line. Coordinates for the "man in the moon":

Head: (–2,4) (2,2) (4,–2) (3,–4) (1,–4) (–2,–2) (–4,2)

Eyes: (–2,2) (0,2)

Nose: (0,0)

Mouth: (–1,–1) (1,–2) (2,–1)

Left Ear: (–4,2) (–5,2) (–4,1)

Right Ear: (2,2) (3,2) (3,1)

To play the game of caricatures:

a) One student draws a picture on a grid of his choosing and without showing the picture supplies only the coordinates of key points in the picture.

b) The other students on self-made grids, using any desired angle for the axes, plot the coordinates on their own grid and sketch in the picture.

2. Operational Checkers (Optional)

This game is played by two players on a finite set of lattice points. For example:

(0,2)	(1,2)	(2,2)
(0,1)	(1,1)	(2,1)
(0,0)	(1,0)	(2,0)

Figure 7.19

You will need to use the arithmetic of $(Z_3, +)$ so we will list the necessary facts: $0 + 0 = 0$: $0 + 1 = 1$: $0 + 2 = 2$: $1 + 1 = 2$: $1 + 2 = 0$: $2 + 2 = 1$: and the commutative property will provide the other basic facts.

1) One player has red checkers and the other has black checkers. A coin is tossed to determine who starts.

2) The first player places a checker on any point that he wishes:

3) The second player may then place a checker on any uncovered point and another point with coordinates obtained by adding the corresponding coordinates of the last two points covered. The addition to be used is that for $(Z_3, +)$.

4) On each subsequent play, if the players opponent has just placed a checker on (c,d), then the player may not only cover any uncovered point (a, b) but also (a + c, b + d). If the point is already covered by his opponent, the player replaces it with one of his own. For example, if one player has just covered (2,1), the other player may cover (2,2) and also (2 + 2, 1 + 2) which is (1,0).

5) The game ends when all points are covered. The winner is the player with the most points covered. As you play the game you will see that it involves several interesting strategies.

7.16 Sets of Lattice Points and Mappings of Z into Z.

You are familiar with many types of mappings from Chapter 3. An important use of lattice points is the representation of mappings of Z into Z.

The diagram below displays some of the assignments made by f: $x \longrightarrow 2x$ where x is a member of Z.

$$
\begin{array}{lccccccc}
\text{Domain} & \{\ldots -3 & -2 & -1 & 0 & 1 & 2 & 3 \ldots\} \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{Range} & \{\ldots -6 & -4 & -2 & 0 & 2 & 4 & 6 \ldots\}
\end{array}
$$

The pairs associated by the mapping can also be displayed as a subset of Z X Z.

{... (-3,-6), (-2,-4), (-1,-2), (0,0), (1,2), (2,4), (3,6), ...}

This subset can be graphed:

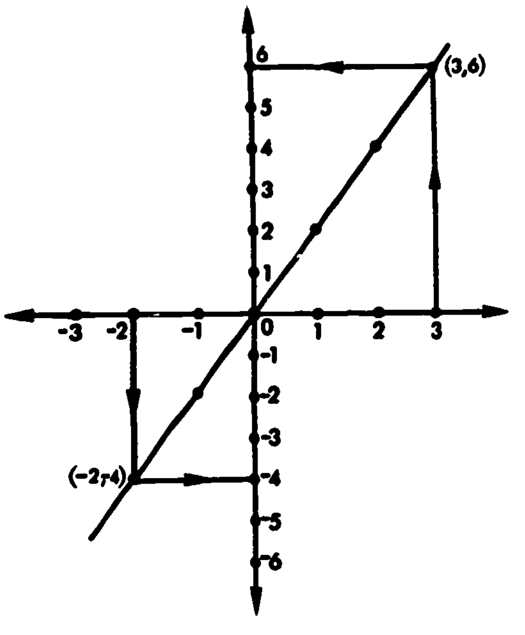


Figure 7.20

In this particular mapping we see that $(x,y) \longrightarrow (x,2x)$ or that $y = 2x$. The arrow from 3 on the x-axis to the point (3,6) and the arrow from the point (3,6) to the point 6 on the y-axis illustrate a geometric method of using the graph to find the integer on the y-axis assigned to a particular integer selected from the x-axis.

Select some other integers from the domain of the mapping illustrated in Figure 7.20 and trace the path from the related point on the x-axis to the point in the graph and then over to the corresponding members of the range on the y-axis.

Which axis contains the graph of the domain of a mapping?

Which axis contains the graph of the range of a mapping?

There is also an algebraic method of finding the image of an integer in the domain of a mapping which is very valuable in graphing.

Example: Graph the mapping which assigns y in the range to x in the domain under the condition $xy = 12$.

(1) Select an integer from the domain, say 2.

(2) Substitute 2 for x in $xy = 12$; then $2y = 12$.

(3) If $2y = 12$, then $y = 6$.

(4) We see that 2 maps onto 6 and (2,6) is in the graph.

In this manner we can find other pairs and record them in a table:

Domain	Range
2	6
3	4
4	
6	
-2	
-3	
-4	

Figure 7.21

Copy and complete the above table. Draw axes on a piece of graph paper and circle the points obtained from the table.

7.17 Exercises

1. Make a table like that in Figure 7.21 for each of the following open sentences:

   (a) $y = x^2$

   (b) $y = 2x + 1$

   (c) $y = x^2 + 1$

   (d) $y = 2x - 1$

   (e) If x is even, $y = 9$ and if x is odd, $y = 1$.

2. Use the tables that you constructed in Exercise 1 to circle the points in the graph of each condition. Use graph paper and make a separate pair of axes for each graph.

7.18 Lattice points for Z X Z X Z

If Z represents the set of integers, and Z X Z represents the set of all ordered pairs of integers, what do you think Z X Z X Z represents?

You have seen that Z may be associated with a set of points on a line and that Z X Z may be associated with a set of points in a plane. The set of all ordered triples of integers may be associated with a set of points in space (3-dimensional).

Suppose that you wish to meet a friend in an office building on the corner of some avenue and street. You not only need to know the number of the street and the number of the avenue, but also the number of the floor in the office building.

The longitude and latitude of an airplane at any instant is not sufficient to determine its position. You also need to know its altitude.

Although two directed numbers are sufficient to get you to the point beneath which the treasure is buried, you don't know where the treasure is until you know how deeply it is buried.

In each of the three examples above, it is necessary to have a triple of numbers to locate an object in space. In a corresponding way, we associate each point in a three-dimensional set of lattice points with an ordered triple of integers. In this case we have three axes instead of two and each point has three coordinates.

Figure 7.22 illustrates the assignment of coordinates to certain points in space. Study the diagram and see if you can discover how each triple, (x,y,z), was assigned.

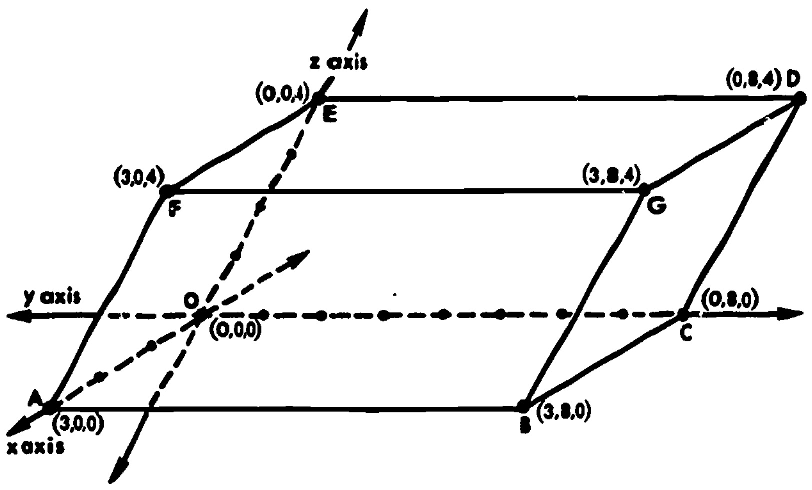


Figure 7.22

The geometric figure with vertices OABCDEFG has six faces each of which is a parallelogram. It is called a *parallelepiped.*

7.19 Exercises

1. (a) Name the six faces of the parallelepiped using the letters that name the vertices.
   (b) How many of the parallelograms have O as a vertex?
   (c) Try to draw the parallelepiped that has O as a vertex for three of its faces and has the point (2,3,4) as the other end of the diagonal from O.
2. (a) Using three pieces of cardboard, try to construct model of three planes so that any pair of planes has a line in common, but all three have only one point in common.

7.20 Translations in Z X Z

In earlier chapters you learned about translations as a special kind of mapping. You also learned that the set of translations in a line, as represented by directed numbers, with the operation "followed by" has the properties of a commutative group.

In this chapter we will be chiefly interested in translations of a set of lattice points into itself in terms of coordinates.

We will designate the image of point P in a mapping by "P′" (read: p-prime). If the coordinates of P are (x,y), then the coordinates of P′ are (x′,y′).

Translations "move" every point in the lattice the same distance in the same direction.

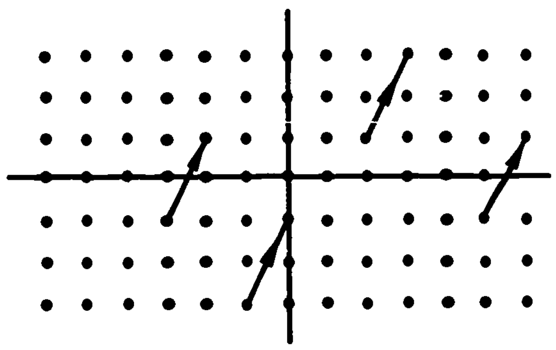


Figure 7.23

The diagram in figure 7.23 shows the effect of a certain translation on four points:

$$(-4,-1) \longrightarrow (-3,1)$$
$$(-1,-3) \longrightarrow (0,-1)$$
$$(5,-1) \longrightarrow (6,1)$$
$$(3,1) \longrightarrow (4,3)$$

Questions: In each case, by how much did the 1st coordinate increase?

By how much did the second coordinate increase?

What is the image of the following points in the same translation?

(a) (2,3) (b) (6,-2) (c) (-1,2) (d) (0,0)

The above translation may be defined by:

$$(x,y) \longrightarrow (x+1,y+2) \quad \text{or by} \quad T_{1,2}.$$

This indicates that the translation adds 1 to the first coordinate of each point and 2 to the second coordinate.

Any translation of Z X Z may be designated by the form:

$$(x,y) \longrightarrow (x+a,y+b) \quad \text{or by} \quad T_{a,b}$$

where a and b are any integers.

What would be the effect of the translation $T_{0,0}$?

Since $T_{0,0}$ or $(x,y) \longrightarrow (x+0,y+0)$ maps every point onto itself it is called the <u>identity</u> translation.

You are familiar with the composition of mappings. In connection with translations of a set of lattice points the composition of $T_{a,b}$ with $T_{c,d}$ can be expressed as:

$$T_{a,b} \circ T_{c,d} = T_{c+a,d+b}$$

The symbol "o" in the definition above can be read "with" or "following" since the translation on the right of the sign translates first. The effect of the above composition of translations on a point (x,y) is:

$$(x,y) \longrightarrow (x+c+a,\ y+d+b)$$

If you placed a disk on a lattice point, the composition $T_{2,3} \circ T_{-4,-1}$ would tell you to first move the disk 4 points to the left and 1 down and follow this 2 to the right and 3 up. Since $T_{2,3} \circ T_{-4,-1} = T_{-2,2}$ this should be the same as moving 2 to the left and 2 up. Figure 7.24 illustrates this by showing the effect on (0,0):

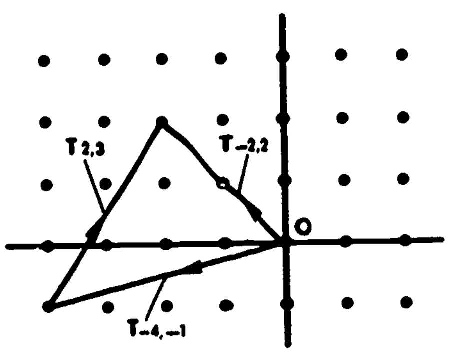


Figure 7.24

7.21 Exercises

1. Find the composites of the following pairs of translations:

   (a) $T_{-5,3} \circ T_{5,-3}$ (b) $T_{4,-2} \circ T_{-4,2}$

   (c) $T_{a,b} \circ T_{-a,-b}$

   If the composite of two translations is the identity translation, each is called the inverse of the other.

2. Use the commutative property for addition of integers to show that $T_{a,b} \circ T_{c,d} = T_{c,d} \circ T_{a,b}$.
3. What property does Exercise 2 demonstrate for composition of translations?
4. Use a property of integers to show the following:

   $T_{a,b} \circ (T_{c,d} \circ T_{e,f}) = (T_{a,b} \circ T_{c,d}) \circ T_{e,f}$
5. What property of composition of translations is demonstrated in Exercise 4?
6. Draw the following parallelogram on graph paper:

   (−3,−1), (0,3), (7,3), (4,−1)
7. Verify with a ruler that the midpoint of the diagonals of the parallelogram in Exercise 6 is (2,1).
8. Find the images of the points on the parallelogram in Exercise 6 under the translation $T_{-2,-1}$, that is map each (x,y) onto (x−2,y−1).
9. Do the image points you found in Exercise 8 form another parallelogram?
10. Verify with a ruler that the midpoint of the image parallelogram's diagonals is the image of the midpoint of the diagonals of the parallelogram in Exercise 6.
11. Is the image parallelogram of the previous exercise the same size and shape as the one of which it is the image?

When we say that a mapping preserves some property, we mean that if a set of points has that property, the image also has that property.

12. On the basis of the translation that you experimented with in the previous exercises, would you say that translation preserves:

    (a) Size and shape? (c) Midpoints?

    (b) Parallelism? (d) Lines?

7.22 Dilations in Z X Z

Figure 7.25 shows graphically what happens to a set of points under dilation:

A dilation of Z X Z is a mapping designated by:

$(x,y) \longrightarrow (ax,ay)$ or D_a, for any non-zero integer a.

If a is less than zero this is sometimes called a dilation with point symmetry in the origin.

In the dilation of the above picture a = 2 or D_2 maps (x,y) onto (2x,2y). An equivalent way to say this is that distances between pairs of points in the image are twice as great as the distances between the corresponding pairs of points in the first picture. If the dilation had been D_{-2}, the image would have been the same size but would have been upside down in the third quadrant with his nose still against the y-axis but 6 units below the origin.

Exercise: Dilate the original picture by a factor of −2. Then the mapping is (x,y) (−2x,−2y).

You will see him increase in size and stand on his head!

In any dilation both coordinates of each point are multiplied by the same number. We will refer to this number as "a" in the following questions:

(a) What happens to points in a dilation if a = 1?

(b) What happens to points in a dilation if a = −1?

(c) If we should allow a to be zero, onto which point would each point map?

(d) What happens to each point in a set of points if a = 3? −3?

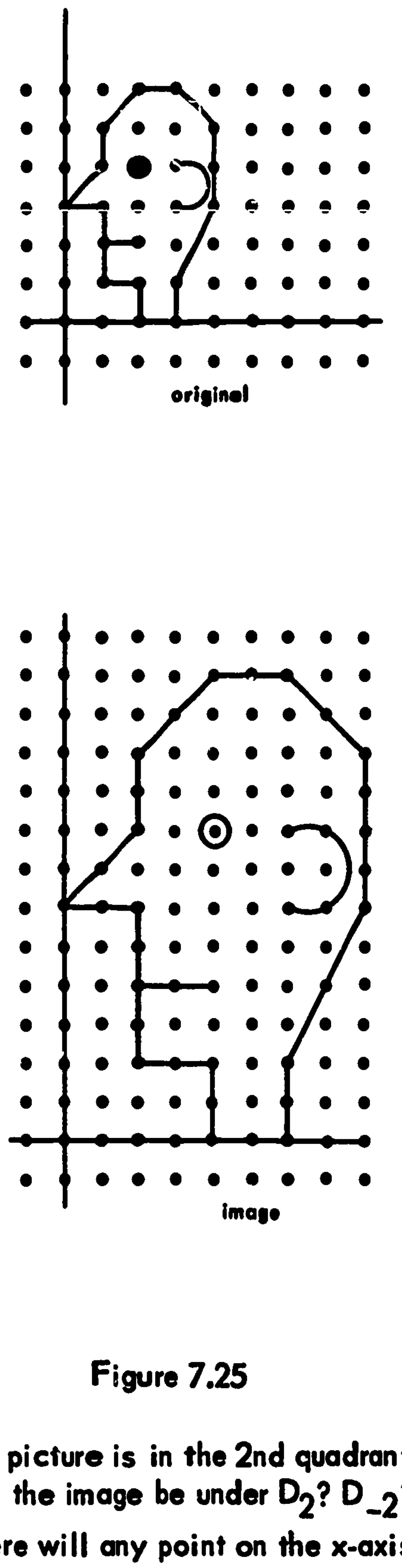


Figure 7.25

(e) If a picture is in the 2nd quadrant, where will the image be under D_2? D_{-2}?

(f) Where will any point on the x-axis be mapped by any dilation of the form $(x,y) \longrightarrow (ax,ay)$? Where will a point on the y-axis be mapped?

7.23 Exercises

1. Use the dilation D_3 to graph the following points and their images:

   (–3,–1) (0,3) (7,3) (4,–1)

2. Answer the following questions about the figure in Exercise 1 and its image:

   (a) What kind of geometric figure is the original figure?

   (b) Is the image the same size as the original? Shape?

   (c) Are the lines in the image parallel to the lines in the original?

   (d) Are midpoints preserved by dilation in this case?

3. The composition of dilations may be represented as

   $D_b \circ D_a = D_{ab}$ where D_a dilates first.

   (a) Which dilation maps every point onto itself?

   (b) Is composition of dilations commutative? Associative?

   (c) Which two dilations are the only ones that have inverses in Z X Z?

7.24 Some Additional Mappings in Z X Z

By now you should have some skill in finding images if you are given a point (coordinates) and a rule for finding the image. For each of the mappings below, find the images of the following points which determine a parallelogram and the midpoint of its diagonals. Then answer questions (a)—(g).

Parallelogram (–3,–1), (0,3), (7,3), (4,–1)

Midpoint of diagonals (2,1)

(a) First use graph paper to graph the figure and its image.

(b) Is the image another parallelogram?

(c) Is the image of the midpoint the midpoint in the image parallelogram?

(d) Does the image have the shape of the original? The size?

(e) If the vertices of the parallelogram are named ABCD clockwise in that order are A′, B′, C′, D′ also in clockwise order?

(f) For each mapping try a special case of composition with a mapping of the same kind.

(g) Try composing pairs of mappings of different kinds.

Mappings:

(1) $(x,y) \longrightarrow (x,-y)$

(2) $(x,y) \longrightarrow (-x,y)$

(3) $(x,y) \longrightarrow (y,x)$

(4) $(x,y) \longrightarrow (y,-x)$

(5) $(x,y) \longrightarrow (x+3,-y)$

(6) $(x,y) \longrightarrow (x+2y,y)$

7.25 Summary

1. The assignment of ordered pairs of integers to lattice points in a plane involves:

(a) Assignment of integers to equally spaced points on each of two intersecting lines called axes;

(b) Parallel projection to assign pairs of integers, one from each axis, to lattice points in the plane of the axes;

(c) Geometric properties of parallel lines and intersecting lines to show that the assignment in (b) is a one-to-one correspondence.

2. The set of all ordered pairs of integers is named Z X Z, and the two integers assigned to a point are called coordinates of the point.

3. Conditions for coordinates of a point, such as, "the sum of the coordinates is 3", are expressed by open sentences, such as "$x + y = 3$". The set of ordered pairs, each of which satisfies the condition, is called the solution set of the condition (or the open sentence). The set of lattice points, that have these pairs for coordinates, is the graph of the condition.

4. Compound conditions may be expressed by connecting two open sentences with "and". The connective "or" can also be used. A pair of integers satisfies an "and" condition, if it satisfies both connected conditions and satisfies an "or" condition, if it satisfies either.

5. The absolute value of an integer is defined:

$$|x| = x, \text{ if } x \geq 0$$
$$|x| = -x, \text{ if } x < 0$$

6. The idea of a coordinate system in a plane may be extended to space by assigning number triples to points.

7. Translations of Z X Z are expressed by:

$(x,y) \longrightarrow (x+a, y+b)$, where a and b are integers.

8. Dilations of Z X Z are expressed by:

$(x,y) \longrightarrow (ax,ay)$, where a is a non-zero integer.

9. All of the mappings of Z X Z presented in this chapter preserve lines, parallels, and midpoints.

7.26 Review Exercises

1. List five ordered pairs of integers that satisfy the condition:

(a) $x + 2y = 5$ (b) $x = 2y$ (c) $y = |x| - 2$

(d) $|x| + |y| = 3$ (e) $xy = 24$

2. Translate the following conditions into open sentences:

(a) Two times the first coordinate minus three times the second coordinate is equal to seven.

(b) The first coordinate is three less that two times the absolute value of the second coordinate.

(c) The first coordinate is greater than zero and the second coordinate is less than two.

3. Translate the following open sentences into words:

(a) $y = x^2 - 2$ (b) $|x + y| = 5$ (c) $y > 2$ or $x < 3$.

4. Tabulate the solution set of the following:

(a) $x + y = 5$ and $x - y = 3$ (b) $y = x^2$ and $x = -1$.

5. Graph the following:

(a) $y = 2x - 1$ (b) $y = -3x$ (c) $x > 0$ and $y = 0$.

6. Which quadrant or quadrants contain the points whose coordinates satisfy the following:

(a) $x = 2$ and $y > 0$ (b) (x,y) is not on either axis.

(c) $y < -5$ and $x < -6$ (c) $x = -10$ and $y = 23$

7. Draw a pair of axes on a piece of graph paper and circle the following points:

(6,11) (6,1) (11,6) (1,6) (9,10) (3,10) (3,2) (9,2) (10,9) (10,3) (2,3) (2,9)

8. Find the image of each point in Exercise 7 for the following mappings and circle the image points

(a) $(x,y) \longrightarrow (x,-y)$ (c) $(x,y) \longrightarrow (-x,-y)$

(b) $(x,y) \longrightarrow (-x,y)$ (d) $(x,y) \longrightarrow (y,x)$

9. Find the image of the following triangle and draw the image triangle on a piece of graph paper for each of the following mappings:

(0,0), (0,5), (2,0),

(a) $(x,y) \longrightarrow (2x,2y)$ (b) $(x,y) \longrightarrow (-2x,-2y)$

(c) $(x,y) \longrightarrow (-2x,2y)$ (d) $(x,y) \longrightarrow (2x,-2y)$

10. Find the image of the parallelogram, (0,0) (0,3) (4,3) (4,0), for each of the following mappings and draw the image on a piece of graph paper:

(a) $(x,y) \longrightarrow (x+3,y+4)$ (b) $(x,y) \longrightarrow (x+2y,y)$

(c) $(x,y) \longrightarrow (x+5,-y)$ (d) $(x,y) \longrightarrow (x,0)$

CHAPTER 8 SETS AND RELATIONS

8.1 Sets

Our everyday speech abounds with collective nouns such as herd, company, swarm, class, litter, collection, bunch, etc. Examples which use these collective nouns would include the following: a herd of cattle, a company of soldiers, a swarm of bees, a class of students, a litter of kittens, a collection of stamps, a bunch of bananas.

It is also possible to find examples which use collective nouns which may be unfamiliar to you such as the following: a gam of whales, a pod of seals, a glitter of butterflies, a singular of boars, a gaggle of geese, a hutch of rabbits, an army of ants, a murmuration of starlings, a jubilation of skylarks, and a pride of lions.

In each of the above examples we see how a word, such as herd, class, pride, etc., is used to denote a collection of several objects assembled together and thought of as a unit. Each of the above collections is said to be well-defined. By this we mean that we can determine if a given object does or does not belong to the specific collection being considered.

In mathematics we use the collective noun <u>set</u> to indicate any well-defined collection. The objects in sets can be literally anything: numbers, points, lines, people, letters, cities, etc.. These objects in sets are called the elements or members of the set. Terms such as "set" and "element" are part of the basic language used in the study of all branches of mathematics. Thus, in this chapter, we will concentrate on terms and concepts dealing with sets and relations between sets.

Let us list ten particular examples of sets.

Example 1: The numbers 1, 2, 3, 4, 5, and 6.

Example 2: The solution set of the open sentence $2 + 5 = x$ in (W, +).

Example 3: The vowels in the English alphabet: a, e, i, o, and u.

Example 4: The states in the U.S.A. whose names begin with the letter "M".

Example 5: The numbers 1, 2, 3, 4, 6, 8, 12, and 24.

Example 6: The states in the U.S.A. for which the names of both the state and its capital city begin with the same letter.

Example 7: The numbers −2, −1, 0, 1, and 2.

Example 8: The set of whole numbers which are both even and odd.

Example 9: The numbers 1, 3, and 5.

Example 10: The outcome set for the tossing of a die.

Notice that the sets in the odd numbered examples above are defined by actually listing the elements in the set; and the sets in the even numbered examples are defined by stating properties which can be used to determine if a particular object is or is not an element of the set.

Sets will usually be denoted by capital letters,

$$A, \quad B, \quad X, \quad Y \ldots$$

Recall that we used "W" to denote the set of whole numbers and "Z" to denote the set of integers. The elements in sets will usually be denoted by lower case letters

$$a, \quad b, \quad x, \quad y, \ldots$$

There are essentially two ways to specify a particular set. One way, if it is possible, is to actually list the elements in a set. For example,

$$A = \{0, 1, 2, 3\}$$

denotes the set A whose elements are the whole numbers 0, 1, 2, and 3. Note that the elements are separated by commas and enclosed in braces { }. The second way to specify a set is by stating those properties which determine or characterize the elements in the set. For example,

$$A = \{ x : x \text{ is a whole number}, x < 4 \}$$

which is read, "A is the set of all x such that x is a whole number and x is less than 4."

Note: A letter, here x, is used to denote an arbitrary element of the set; the colon ":" is read as "such that"; the comma is read as "and."

If an object x is an element of a set A, i.e., A contains x as one of its elements, then we write

$$x \in A$$

This can also be read "x is a member of A", or "x is in A", or "x belongs to A". To indicate that "x is not an element of set A" we write

$$x \notin A.$$

Thus, in the set A given above we have

$$0 \in A, 1 \in A, 2 \in A, 3 \in A, \text{ and } 4 \notin A.$$

Let us rewrite the Examples 1.10 given earlier in order to illustrate the above remarks and notation. We shall denote the sets by $A_1, A_2, A_3, \ldots, A_{10}$ respectively.

Example 1′: $A_1 = \{1, 2, 3, 4, 5, 6\}$

Example 2′: $A_2 = \{x : x = 2 + 5, x \in W\}$

Example 3′: $A_3 = \{a, e, i, o, u\}$

Example 4′: $A_4 = \{x$: x is a state beginning with the letter "M"$\}$

Example 5′: $A_5 = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Example 6′: $A_6 = \{x$: x is a state whose first letter is the same as the first letter of its capitol city$\}$

Example 7′: $A_7 = \{-2, -1, 0, 1, 2\}$

Example 8′: $A_8 = \{x : x \in W, x \text{ is even}, x \text{ is odd}\}$

Example 9′: $A_9 = \{1, 3, 5\}$

Example 10′: $A_{10} = \{x$: x is an outcome of a toss of a die$\}$

8.2 Exercises

1. Can you find the eight elements in the A_4?
2. Can you find the four elements in the set A_6?
3. What relationship exists between the sets A_1 and A_{10}?
4. What relationship exists between the sets A_9 and A_1?
5. List the elements in the sets
   a) A_2
   b) A_8
6. Can you denote the following sets by stating a property which determines or characterizes the elements in the set?
   a) A_5
   b) A_7
   c) A_9
7. List four essentially different sets that you have studied in previous chapters in this book.
8. What special name do we give to the set defined in Example 8′ above?
9. Explain why or why not the following are true.
   a) $7 \in A_{10}$
   b) Delaware is an element of set A_6
   c) $0 \in A_8$
   d) $x \notin A_3$
10. Can you state a property that is true of all the sets $A_1 - A_{10}$?

8.3 Set Equality; Subsets

Let $A = \{0, 1, 2, 3\}$

and $B = \{1, 0, 3, 2\}$

If we examine the above we find that both sets A and B contain precisely the same elements although they are not listed in the same order. The sameness between the two sets A and B we will indicate by writing

$$A = B$$

and which we read as "set A is equal to set B". Equality between two sets means that the two sets contain precisely the same elements and that we do not have two sets but only one.

In general, if A denotes a set, and B denotes a set, then for sets A and B, the statement

$$A = B$$

means that A and B denote the same set. If sets A and B are not equal then we shall write

$$A \neq B.$$

Example 1: If $X = \{0, 1\}$ and $Y = \{x : x \in W, x < 2\}$, then we have that $X = Y$

Example 2: If $V = \{a, e, i, o, u\}$ and $Y = \{u, e, a, o, i\}$, then $V = Y$ Note that the order in which the elements are listed is immaterial

Example 3: If $V = \{a, e, i, o, u\}$ and $X = \{x$: x is a letter in the english alphabet$\}$ then we have that

$$V \neq X$$

In Example 3 above we notice that there is a relationship, other than equality, between sets V and X. It is clear that every element of set V is also an element of set X. We say that set V is a <u>subset</u> of set X or that set V is <u>contained in</u> set X. We denote the relation "is a subset of" by the symbol "$\subseteq$". Thus, in Example 3, $V \subseteq X$.

Definition: Set A is called a subset of set B, denoted by $A \subseteq B$, if and only if the sets A and B have the property that every element of set A is an element of set B.

Notice that the above definition means that if $A \subseteq B$ and $x \in A$, then $x \in B$.

Example 1: Let $A = \{1\}$, $B = \{0, 1, 2\}$, $C = \{3, 4, 5, 6\}$, and $D = \{0, 1, 2, 3, 4, 5\}$ Then we see that

$A \subseteq B$, $B \subseteq D$, $A \subseteq D$.

Note that C is not a subset of D because $6 \in C$ but $6 \notin D$.

Example 2: Let $X = \{a, b, c\}$ and $Y = \{c, a, b\}$. We see that $X \subseteq Y$ because every element of X is an element of Y. Furthermore $Y \subseteq X$.

Notice in Example 1 why C is <u>not</u> a subset of D. It illustrates the following general remark:

Remark 1: If set A is not a subset of set B, then set A contains at least one element that is not contained in set B.

Notice also that Example 2 shows that $A \subseteq B$ does not exclude the possibility that $A = B$. In fact, we can make the following general remark concerning how equality of sets is related to the idea of subset:

Remark 2: Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

~~Let us illustrate the above statement:~~

If $A = \{0, 1, 2, 3\}$ and $B = \{1, 0, 3, 2\}$ then clearly $A \subseteq B$ because every element in set A is also an element in set B. Also $B \subseteq A$ because every element in set B is also an element in set A. Thus, we conclude that $A = B$.

From the above we see that every set has at least one subset, namely, itself. In fact, we can make the following general remark:

Remark 3: If A is any set, then $A \subseteq A$.

We can examine a given set to see what subsets it contains. For example, what subsets are contained in the set $A = \{a, b\}$? From the above remark we see that A contains itself as a subset. Thus $\{a, b\} \subseteq A$, or equivalently, $\{a, b\} \subseteq \{a, b\}$. Also it is clear that set A contains two subsets each of which contains a single element. That is

$\{a\} \subseteq A$ and $\{b\} \subseteq A$.

Next let us consider the empty set as a possible subset of A. We are asking if

$\emptyset \subseteq A$, or equivalently, if $\{\ \} \subseteq \{a,b\}$?

If we assume that $\emptyset$ is not a subset of A, then Remark 1, given earlier, implies that $\emptyset$ contains at least one element that is not an element of A. But $\emptyset$ contains no such element since by definition $\emptyset$ contains no elements. Thus we cannot say that $\emptyset$ is not a subset of A., i.e., $\emptyset$ is a subset of A. Since the above argument would apply to any set A, we make the following remark:

Remark 4: If A is any set, then $\emptyset \subseteq A$.

Note that the set $A = \{a, b\}$ contains exactly four subsets, namely $\{a\}$, $\{b\}$, $\{a,b\}$ and $\emptyset$.

Of these four subsets of A we shall say that $\{a\}$, $\{b\}$, and $\emptyset$ are proper subsets of A and that $\{a, b\}$ is not a proper subset of A. Note that proper subsets of a set do not contain all the elements of the given set. In general we have the following definition:

Definition: A is a proper subset of B, denoted by $A \subset B$, if and only if $A \subseteq B$ and $A \neq B$.

"$A \subset B$" is read "A is a proper subset of B" or "A is properly contained in B."

Example 1: Let $K = \{-1, 0, 1\}$. Then $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$ are each proper subsets of K.

Note $\{-1, 0, 1\}$ is a subset of K but not a proper subset of K.

Example 2: Let X be any set except the empty set. Then, because we know that $\emptyset \subseteq X$ by Remark 4 and because we are given that $\emptyset \neq X$, we conclude that $\emptyset$ is a proper subset of X, that is $\emptyset \subset X$.

8.4 Exercises

1. Let $G = \{0, 1, 3, 7\}$ and $H = \{7, 1, 0, 3\}$. Explain why or why not $G = H$
2. If $G = \{0, 1, 3, 7\}$ and $L = \{x : x \in W, x < 10\}$ then explain why
   (a) $G \subseteq L$ (c) $G \subset L$
   (b) $G \neq L$ (d) $\emptyset \subset G$
3. Let B = {Tom, Dick, Harry}, G = {Judy, Joan} R = {Tom, Joan, Harry, Judy}.
   (a) Explain why or why not B is a subset of R.
   (b) Explain why or why not G is a subset of R.
4. Let $E = \{x : x \in W, x \text{ is even}\}$ and $P = \{x : x \text{ is a positive power of } 2\}$, i.e., $P = \{2, 4, 8, 16, \ldots\}$
   Explain why the following are or are not true:
   (a) $P \subseteq E$ (e) $100 \in E$
   (b) $P \subset E$ (f) $100 \in P$
   (c) $P = E$ (g) $E \subseteq P$
   (d) $0 \in E$ (h) $\emptyset \subset P$
5. Let $A = \{a\}$
   (a) List all of A's subsets.
   (b) List all of A's proper subsets.
6. Let $B = \{a, b, c\}$
   a) List all of B's subsets.
   b) List all of B's proper subsets.
7. Using the data obtained in Exercise 5 and 6 above can you make a conjecture concerning:
   a) the number of subsets in a set containing 4 elements?
   b) the number of proper subsets in a set containing 4 elements?
   c) the number of subsets in a set containing 5 elements?
   d) the number of proper subsets in a set containing 5 elements?
   e) the number of subsets in a set containing n elements?
   f) the number of proper subsets in a set containing n elements?
8. What can we conclude if we know that A is a subset of B but that B is not a subset of A?

9. Tom, Dick, Harry, Judy, and Ann agree that when they line up to be photographed the boys and girls will alternate. List the set of all possible ways of lining up.

10. In Exercise 9, list the following subsets:

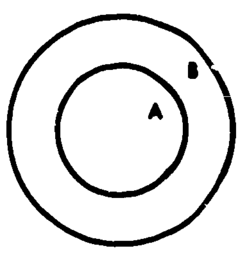
 a) The set in which Tom and Ann are next to each other.

 b) The set in which Judy is between Harry and Dick.

 c) The set in which Harry is in the middle.

 d) The set in which Judy is in the middle.

Note: When working with subsets it is often helpful to use a drawing such as



to indicate that A is a subset of B. Disks, that is, circles and their interiors are used to represent the sets A and B. The drawing shows that all of set A is contained in set B, i.e., $A \subseteq B$.

11. What conclusions, if any, can you draw from the following:

 a) $X \subset Y$ and $Y \subset Z$?

 b) $R \subset S$ and $T \subset R$?

 c) $M \subseteq N$ and $N \subset Q$?

 d) $X \subset G$, $Y \subset T$, and $T \subset X$?

 e) $A \subseteq Q$, $Q \subseteq R$, and $R \subseteq A$?

 f) $P \subset Q$ and $R \subset Q$?

12. Let $A = \{p, q, r\}$. Explain why the following are correct or incorrect.

 a) $p \in A$ d) $A \subset A$

 b) $p \subset A$ e) $\{p\} \in A$

 c) $\{p\} \subset A$ f) $\emptyset \subset A$

13. Which of the following sets are equal?

 a) $\{x :$ x is a letter in the word "follow"$\}$

 b) $\{x :$ x is a letter in the word "wolf"$\}$

 c) the set of letters in the word "flow"

14. Explain why the sets $\emptyset$ and $\{\emptyset\}$ are different sets.

15. Let $X \subset Y$ and $Y \subset Z$. Assume $x \in X$, $y \in Y$, $z \in Z$, and also assume $p \notin X$, $q \notin Y$, $r \notin Z$. Which of the following must be true? Explain

 a) $x \in Z$ c) $z \notin X$ e) $q \notin X$

 b) $y \in X$ d) $p \in Y$ f) $r \notin X$

8.5 Universal Set, Unions, Intersections, Complements

In order to avoid certain logical difficulties, we will assume that in a given discussion the sets being considered are subsets of a set S, called the universal set. We have already seen situations where the idea of a universal set played an important role. For example, in finding solution sets for open sentences we have seen that results depend on the domain or universal set considered.

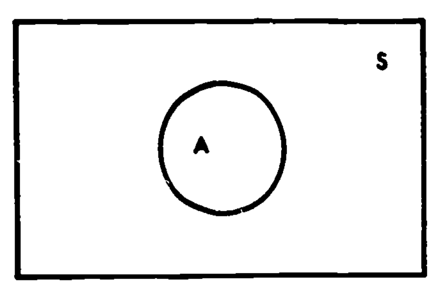
The solution set of the open sentence

$$3 + x = 2$$

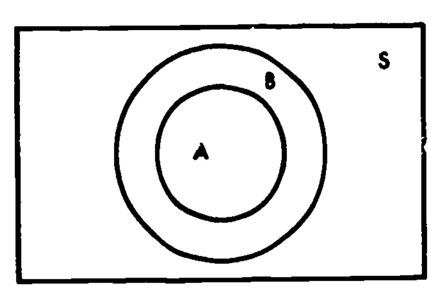
is $\{-1\}$ if the universal set considered is the set Z whereas it is $\emptyset$ if the universal set is set W.

In order to help visualize our work with sets we shall draw diagrams, called Venn diagrams, which illustrate them. Here we represent a set by a simple plane region, usually bounded by a circle. We shall indicate the universal set S by a plane region bounded by a rectangle.

Example 1: To indicate that a set A is a subset of a universal set S we have



Example 2: To indicate that A is a proper subset of B where both A and B are subsets of a universal set S we have



In earlier chapters you considered operations which assigned new numbers to given ordered pairs of numbers. Next we shall consider how new sets can be formed from given sets. There are two important binary operations that we shall define on ordered pairs of sets. You will find that the sets assigned to pairs of sets A and B by these operations have many uses in your subsequent work. In what follows we assume that the sets A and B are subsets of the universal set S.

If A and B are two sets we shall define a new set called the union of A and B, denoted by "$A \cup B$", as follows:

Definition: $A \cup B$ is the set that contains those and only those elements that belong either to A or to B (or to both) i.e.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

(Notice that here "or" is used in the sense of "and/or.")

Example 1: If $A = \{0, 1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{0, 1, 2, 3, 4, 5\}$.

Example 2: If $V = \{a, e, i, o, u\}$ and $X = \{p, q, r\}$, then $V \cup X = \{a, e, i, o, u, p, q, r\}$.

Example 3: If W is the set of whole numbers and $A = \{0, 1, 2, 3\}$, then $W \cup A = W$.

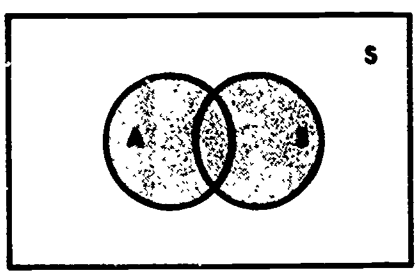
Remark 1: From the definition of $A \cup B$ we can find that

$$A \cup B = B \cup A.$$

Remark 2: Since $A \cup B$ contains all the elements of A and also contains all the elemtnes of B we can conclude that

$$A \subseteq (A \cup B) \text{ and } B \subseteq (A \cup B)$$

In the Venn diagram below we have shaded $A \cup B$, i.e., the region covered by sets A and B.



($A \cup B$ is shaded)

If A and B are two sets we now define a new set called the intersection of A and B, denoted by "$A \cap B$", as follows:

Definition: $A \cap B$ is the set that contains those and only those elements that belong to both A and B, i.e.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Example 1: If $A = \{0, 1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Example 2: If $V = \{q, e, i, o, u\}$ and $X = \{p, q, r\}$, then $V \cap X = \{\ \} = \emptyset$

Example 3: If W is the set of whole numbers and $A = \{0, 1, 2, 3\}$, then $W \cap A = \{0, 1, 2, 3\} = A$.

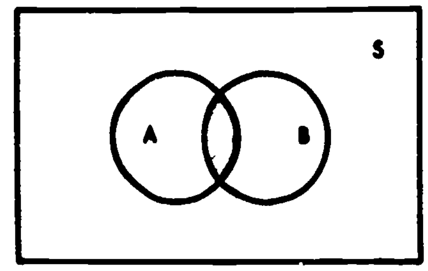
Example 4: If $A = \{0, 1, 2, 3\}$, $B = \{3, 4, 5\}$, and $C = \{0, 3, 5\}$ then $(A \cap B) \cap C = \{3\} \cap \{0, 3, 5\} = \{3\}$.

Remark 1: From the definition of $A \cap B$ we can find that $A \cap B = B \cap A$.

Remark 2: If $A \cap B = \emptyset$, as in Example 2, then we say that A and B are disjoint sets.

In the Venn diagram below we have shaded $A \cap B$ i.e., the area common to both A and B.

Besides obtaining new sets by assigning a new set to a pair of sets it is also useful to define a particular unary operation on every subset of S. If A is a



($A \cap B$ is shaded)

given subset of the universal set S, we can define a new set called the <u>complement</u> of A, denoted by $\overline{A}$, as follows:

Definition: $\overline{A}$ is the set of all elements of S that are not contained in A, i.e.,

$$\overline{A} = \{x : x \in S, x \notin A\}$$

Example 1: If $S = \{0, 1, 2, 3, 4, 5\}$ and $A = \{0, 2\}$, then $\overline{A} = \{1, 3, 4, 5\}$.

Example 2: Let $X = W$, that is, the universal set shall be the set of whole numbers.

Let $E = \{x : x \in W, x \text{ is even}\}$ and $O = \{x : x \in W, x \text{ is odd}\}$.

Then $\overline{E} = O$. That is, the complement of the set of even whole numbers in the set of whole numbers is the set of odd whole numbers. Similarly, $\overline{O} = E$.

Example 3: If $s = \{0, 1, 2, 3, 4, 5\}$, $A = \{0, 2, 4\}$ and $B = \{3, 4\}$, then

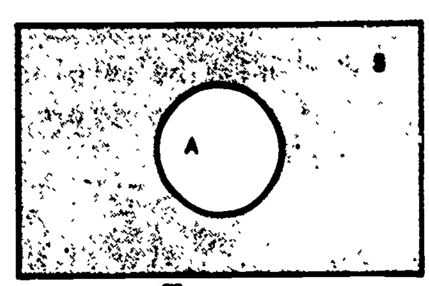
i) $\overline{A} = \{1, 3\}$

ii) $\overline{B} = \{0, 1, 2\}$

iii) Since $A \cap B = \{4\}$ we see that

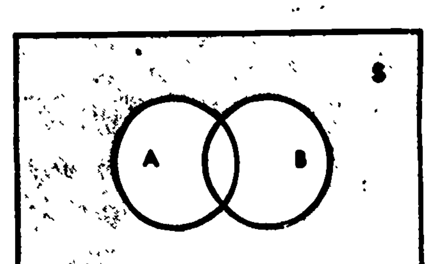
$$\overline{A \cap B} = \overline{\{4\}} = \{0, 1, 2, 3\}$$

The Venn diagram for $\overline{A}$ is given below, i.e., all of S is shaded except A.



($\overline{A}$ is shaded)

The Venn diagram for $\overline{A \cup B}$ is given below. Since $\overline{A \cup B}$ is the set consisting of all elements in S that are not in the set $A \cup B$ we shade all of S except $A \cup B$



($\overline{A \cup B}$ is shaded)

8.6 Exercises

1. Let the universal set S be as follows:

   $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   Further, let $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{2, 3, 5, 7\}$.

Determine the following:

(a) $A \cup B$	(g) $\bar{A}$
(b) $A \cap B$	(h) $\bar{B}$
(c) $A \cup C$	(i) C
(d) $A \cap C$	(j) $\bar{A} \cup \bar{C}$
(e) $B \cup C$	(k) $\overline{A \cap C}$
(f) $B \cap C$	(l) $\bar{S}$

2. Using the sets in Exercise 1 determine the following:

   (a) $(A \cup B) \cup C$ (c) $A \cap (B \cap C)$

   (b) $A \cup (B \cup C)$ (d) $(A \cap B) \cap C$

3. Using the sets in Exercise 1 determine the following:

   (a) $A \cup (B \cap C)$

   (b) $(A \cup B) \cap (A \cap C)$

   (c) $A \cap (B \cup C)$

   (d) $(A \cap B) \cup (A \cap C)$

4. Using the data obtained in the above exercises can you state some conjectures concerning the operations of union and intersection on any sets A, B, and C? Can you offer any further evidence to support your conjectures?

5. Let N be the set of natural numbers, i.e., the set of whole numbers with zero deleted. Let the universal set be W, that is, the set of whole numbers. Determine if the following are true or false. Explain your answers.

   a) $N \cup W = W$ e) $\bar{W} \cup \bar{N} = \emptyset$

   b) $N \cap W = \{0\}$ f) $\bar{W} \cap \bar{N} = \emptyset$

   c) $\bar{N} = \{9\}$ g) $\overline{W \cup N} = \emptyset$

   d) $\bar{W} = \emptyset$ h) $\overline{W \cap N} = \emptyset$

6. Using the definitions of "subset," "intersection," and "union" write out an argument why the following are true:

   a) $(A \cap B) \subseteq A$

   b) $(A \cap B) \subseteq A \cup B$.

7. Using your definitions explain why the following are true: If A is any set in a universal set S, then

   a) $A \cup A = A$ e) $\bar{S} = \emptyset$

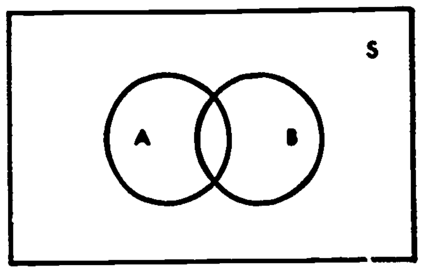
   b) $A \cap A = A$ f) $\bar{\emptyset} = S$

   c) $A \cup \bar{A} = S$ g) $A \cup S = S$

   d) $A \cap \bar{A} = \emptyset$ h) $A \cap \emptyset = \emptyset$

8. If we denote the "complement of the complement of a set A by "$\bar{\bar{A}}$" determine what set $\bar{\bar{A}}$ is equal to.

9. Copy the Venn diagram below and shade in the set represented by $A \cap \bar{B}$.



10. Let us define a new operation, called the difference of A and B, denoted by "$A \setminus B$," as follows:

    $A \setminus B = \{x \mid x \in A, x \notin B\}$

    a) Determine if $A \setminus B = A \cap \bar{B}$.

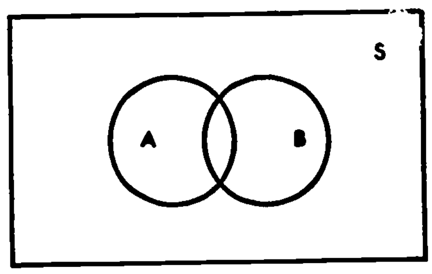
    b) Determine if $A \setminus B = B \setminus A$.

    c) Determine if $(A \setminus B) \subseteq A$

    d) Determine the set represented by the union of $A \setminus B$, $A \cap B$, and $B \setminus A$.

    e) Determine the set represented by the intersection of $A \setminus B$ and $B \setminus A$.

11. Copy the Venn diagram below and shade in the set represented by $(A \cap \bar{B}) \cup (\bar{A} \cap B)$



12. Let us define a new operation called the symmetric difference of A and B, denoted by "$A \Delta B$", as follows

    $A \Delta B = \{x : x \in A, x \in B, x \notin (A \cap B)\}$

    a) Determine if $A \Delta B = (A \cup B) \setminus (A \cap B)$.

    b) Determine if $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

    c) Determine if $A \Delta B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$.

    d) Determine what set is represented by $(A \cup B) \setminus (A \Delta B)$.

8.7 Membership Tables

We shall now show a helpful way of stating our definition of $A \cup B$ by means of a table.

Recall that $x \in (A \cup B)$ if and only if $x \in A$ or $x \in B$ (or both). Thus we have

i) if $x \in A$ and $x \in B$, then $x \in (A \cup B)$

ii) if $x \in A$ and $x \notin B$, then $x \in (A \cup B)$

iii) if $x \notin A$ and $x \in B$, then $x \in (A \cup B)$

iv) if $x \notin A$ and $x \notin B$, then $x \notin (A \cup B)$

Notice that (i) – (iii) above show that $x \in (A \cup B)$ if x is an element of at least one of the sets A or B, and (iv) above shows that $x \notin (A \cup B)$ when and only when $x \notin A$ and $x \notin B$.

The table below is a convenient way of expressing the information given in (i) – (iv) above. We call this the Membership Table for $A \cup B$.

	A	B	$A \cup B$
Row 1:	$\in$	$\in$	$\in$
Row 2:	$\in$	$\notin$	$\in$
Row 3:	$\notin$	$\in$	$\in$
Row 4:	$\notin$	$\notin$	$\notin$

Table 1

Question 1: Can you explain what is meant by the entries in each of the rows in the above table?

Using our definition for $A \cap B$ we obtain the following Membership Table for $A \cap B$

	A	B	$A \cap B$
Row 1:	$\in$	$\in$	$\in$
Row 2:	$\in$	$\notin$	$\notin$
Row 3:	$\notin$	$\in$	$\notin$
Row 4:	$\notin$	$\notin$	$\notin$

Table 2

The above table indicates that $x \in (A \cap B)$ if and only if $x \in A$ and $x \in B$.

Question 2: Can you explain what is meant by the entries in each of the rows in the table above?

Notice that in both of the tables above we had to have four rows in order to consider all possible cases of membership for two sets. It turns out that if we are considering one set then only two rows are required.

For example, the Membership Table for A is

	A	$\overline{A}$
Row 1:	$\in$	$\notin$
Row 2:	$\notin$	$\in$

Table 3

The reason that we have only two rows is easy to understand. We are considering only a single set A. Either $x \in A$ or $x \notin A$. If $x \in A$, then $x \notin \overline{A}$ by definition of $\overline{A}$. This is the information indicated in Row 1. If, on the other hand, $x \notin A$, then $x \in \overline{A}$. Row 2 indicates this fact. Since there is no third membership possibility for x, no further rows are required.

Let us illustrate how membership tables can be used to establish an important property of set. That is

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

[A Law of DeMorgan]

First we form the membership table of $A \cup B$, and then take the complement of $A \cup B$. That is

A	B	$A \cup B$	$\overline{A \cup B}$
$\in$	$\in$	$\in$	$\notin$
$\in$	$\notin$	$\in$	$\notin$
$\notin$	$\in$	$\in$	$\notin$
$\notin$	$\notin$	$\notin$	$\in$

Table 4

Note that the column headed by $\overline{A \cup B}$ makes use of our previous Table 3 for complements.

Then we form the membership table of $\overline{A} \cap \overline{B}$ by working out $\overline{A}$, $\overline{B}$, and then $\overline{A} \cap \overline{B}$. That is

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
$\in$	$\in$	$\notin$	$\notin$	$\notin$
$\in$	$\notin$	$\notin$	$\in$	$\notin$
$\notin$	$\in$	$\in$	$\notin$	$\notin$
$\notin$	$\notin$	$\in$	$\in$	$\in$

Table 5

Note that the columns headed by $\overline{A}$ and $\overline{B}$ make use of our previous Table 3 for complements and the column headed by $\overline{A} \cap \overline{B}$ applies Table 2 to the entries in the previous two columns.

Because the resulting entries in the final columns in Tables 4 and 5 show that

i) if $x \in (\overline{A \cup B})$, then $x \in (\overline{A} \cap \overline{B})$, then we have $(\overline{A \cup B}) \subseteq (\overline{A} \cap \overline{B})$ and

ii) if $x \in (\overline{A} \cap \overline{B})$, then $x \in (\overline{A \cup B})$, then we have $(\overline{A} \cap \overline{B}) \subseteq (\overline{A \cup B})$ we conclude that

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

The information contained in Tables 4 and 5 can be conveniently expressed in one table as follows:

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
$\in$	$\in$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\in$	$\notin$	$\in$	$\notin$	$\notin$	$\in$	$\notin$
$\notin$	$\in$	$\in$	$\notin$	$\in$	$\notin$	$\notin$
$\notin$	$\notin$	$\notin$	$\in$	$\in$	$\in$	$\in$

Table 6

Membership tables can also be used to show that sets are not equal to each other. For example, Table 7 shows that

$$\overline{A \cup B} \neq \overline{A} \cup \overline{B}$$

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$
$\in$	$\in$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\in$	$\notin$	$\in$	$\notin$	$\notin$	$\in$	$\in$
$\notin$	$\in$	$\in$	$\notin$	$\in$	$\notin$	$\in$
$\notin$	$\notin$	$\notin$	$\in$	$\in$	$\in$	$\in$

$\neq$

Table 7

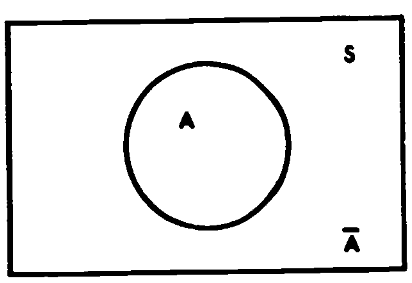
We see that the entries in the columns headed by $\overline{A \cup B}$ and $\overline{A} \cup \overline{B}$ are not identically the same. (The entries in Rows 1 and 4 are the same but the entries in Rows 2 and 3 are not the same.) Thus we conclude that

$$\overline{A \cup B} \neq \overline{A} \cup \overline{B}.$$

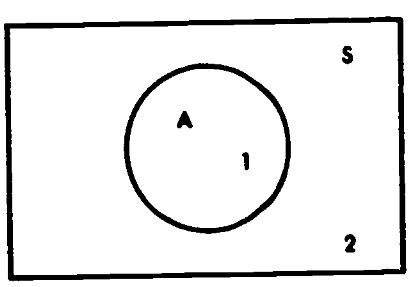
It was shown earlier that if we are considering one set then only two rows are required in the membership table. For example, the table below shows that $\overline{\overline{A}} = A$.

	A	$\overline{A}$	$\overline{\overline{A}}$
Row 1:	$\in$	$\notin$	$\in$
Row 2:	$\notin$	$\in$	$\notin$

Why two and only two rows are required in such tables as Table 8 can be seen if we make use of a Venn diagram. If we reexamine the Venn diagram for sets A and $\overline{A}$, that is

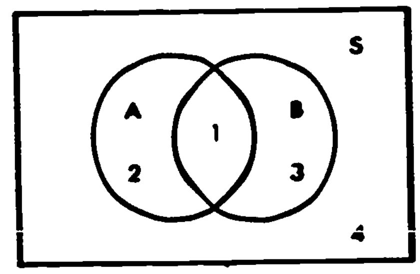


we see that there are two distinct regions determined which we can number, respectively, as 1 and 2. Then we have Regions 1 and 2 corresponding, respectively to sets A and $\overline{A}$ If $x \in A$



or x is an element in Region 1 this corresponds to Row 1 in Table 8. If $x \in \overline{A}$ (or equivalently $x \notin A$) or x is an element in Region 2 this corresponds to Row 2 in Table 8. Since there are two and only two recions to consider there are two and only two corresponding rows to consider.

In a similar way if we are dealing with two sets A and B in an universal set S, such as in Tables 6 and 7 above, we find there are four distinct regions which are disjoint. (i.e. they have no elements in common; see Remark 2 in 8.5). Each of these four disjoint regions corresponds to one of the four rows found in the membership tables involving the two sets A and B. In the Venn diagram below we have numbered these disjoint regions 1, 2, 3, and 4.

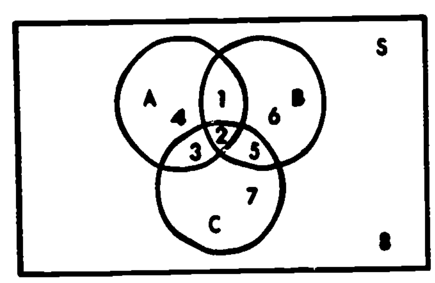


Question 3. Explain why the Regions 1, 2, 3, and 4 in the above Venn diagram correspond, respectively, to the Rows 1, 2, 3, and 4 in the membership table below.

	A	B
Row 1:	$\in$	$\in$
Row 2:	$\in$	$\notin$
Row 3:	$\notin$	$\in$
Row 4:	$\notin$	$\notin$

Table 9

If we draw a Venn diagram to represent any three sets A, B, and C in a universal set S, as below, we find there are eight disjoint regions which we number as indicated below.



Question 4: Explain why the Regions 1, 2, 3, 4, 5, 6, 7 and 8 indicated in the Venn diagram above corresponds, respectively, to the Rows 1, 2, 3, 4, 5, 6, 7 and 8 in the membership table below.

	A	B	C
Row 1:	$\in$	$\in$	$\in$
Row 2:	$\in$	$\in$	$\notin$
Row 3:	$\in$	$\notin$	$\in$
Row 4:	$\in$	$\notin$	$\notin$
Row 5:	$\notin$	$\in$	$\in$
Row 6:	$\notin$	$\in$	$\notin$
Row 7:	$\notin$	$\notin$	$\in$
Row 8:	$\notin$	$\notin$	$\notin$

From the above we see that membership tables involving three sets give rise to eight distinct cases to be considered, each case corresponding to a row in the table. The following partially completed table can be used to determine if $A \cup (B \cap C)$ is or is not, equal to $(A \cup B) \cap (A \cup C)$. (That is, we can determine if "$\cup$" is distributive over "$\cap$".)

A	B	C	$B\cap C$	$A\cup(B\cap C)$	$A\cup B$	$A\cup C$	$(A\cup B)\cap(A\cup C)$
$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
$\in$	$\in$	$\notin$	$\notin$	$\in$	$\in$	$\in$	$\in$
$\in$	$\notin$	$\in$	$\notin$	$\in$	$\in$	$\in$	$\in$
$\in$	$\notin$	$\notin$					
$\notin$	$\in$	$\in$					
$\notin$	$\in$	$\notin$					
$\notin$	$\notin$	$\in$					
$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$

Table 11

8.8 Exercises

1. (a) Copy and complete the following membership table:

A	B	$A\cup B$	$B\cup A$
$\in$	$\in$	$\in$	
$\in$	$\notin$		$\in$
$\notin$	$\in$		
$\notin$	$\notin$		$\notin$

(b) What can you conclude from the table given in 1(a)?

(c) Use a membership table to determine if $A \cap B = B \cap A$.

(d) Use a membership table to determine if

(1) $A \cup A = A$ (2) $A \cap A = A$

(e) How are the tables used in (c) and (d) different?

2. (a) Copy and complete the ~~following~~ membership table

A	B	$A\cap B$	$\overline{A\cup B}$	$\bar{A}$	$\bar{B}$	$\bar{A}\cup\bar{B}$
$\in$	$\in$					
$\in$	$\notin$					
$\notin$	$\in$	$\notin$	$\in$	$\in$	$\notin$	$\in$
$\notin$	$\notin$					

(b) What can you conclude from 2(a)?

3. Use membership tables to determine if the following are or are not, true:

(a) $A \cap (A \cup B) = A$

(b) $\overline{A \cap B} - \bar{A} \cap \bar{B}$

(c) $(A \cap B) \cup (A \cap \bar{B}) = A$

(d) $A \cup (\bar{A} \cap B) = A \cup B$

4. Using membership tables prove that

(a) $A \cup (B \cup C) = (A \cup B) \cup C$

(b) $A \cap (B \cap C) = (A \cap B) \cap C$

5. (a) Copy and complete Table 11 in 8.7

(b) Explain why or why not

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(c) Using a membership table determine if

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. (a) Examine the tables given which involve one set, two sets, and three sets. Can you describe a pattern that was used in assigning "$\in$" and "$\notin$" in the columns headed by the sets being considered?

(b) Can you efficiently set up a table which involves four sets?

(c) How many rows would be necessary in a membership table which considers

(1) Four sets (2) five sets (3) n sets?

7. We define the difference of two sets A and B, denoted by "$A \setminus B$", as the set consisting of all elements of A which are not elements of B. For example if $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $A = \{0, 1, 2, 3\}$, and $B = \{1, 3, 5, 7\}$, then $A \setminus B = \{0, 2\}$ and $B \setminus A = \{5, 7\}$

(a) Explain why the following is the membership table for $A \setminus B$:

A	B	$A\setminus B$
$\in$	$\in$	$\notin$
$\in$	$\notin$	$\in$
$\notin$	$\in$	$\notin$
$\notin$	$\notin$	$\notin$

(b) Using membership tables prove or disprove the following

(1) $A \setminus B = A \cap \bar{B}$

(2) $(A \cup B) \setminus B = A \setminus B$

(3) $(A \setminus B) \cup (B \setminus A) = A \cup B$

(4) $(A \setminus B) \cup (A \cap B) = A$

(c) Using membership tables prove or disprove the following:

(1) $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$

(2) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

8. We can define the symmetric difference of two sets A and B, denoted by "$A \triangle B$", as follows:

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

(a) Construct a membership table to show that

(1) $A \triangle B = (A \cup B) \setminus (A \cap B)$

(2) $(A \triangle B) \cup (A \cap B) = A \cup B$

(3) $(A \cup B) \setminus (A \triangle B) = A \cap B$

(4) $(A \Delta B) \setminus B = A \setminus B$

(5) $A \Delta B = B \Delta A$

(6) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

*9. Recall how Venn diagrams were used to represent one set, two sets, and three sets. Investigate what occurs when four sets are considered. Can you devise a scheme, not necessarily using only circles, which indicates four sets and the proper number of disjoint regions such that each disjoint region corresponds, respectively, to a row of a membership table involving four sets? Write a report of your findings.

8.9 Product Sets: Relations

In earlier parts of this book you have often dealt with the idea of an ordered pair of elements. In many cases you had to distinguish between the pair (a,b) and the pair (b,a). For example, this occurred when you discussed operation, mapping, outcome sets, lattices, etc. In order to stress that the order in which the elements of a pair should be considered is an important idea one of the elements in the pair was designated as the first element or coordinate of the pair and the remaining element was designated as the second element or coordinate of the pair. We shall now make use of the idea of ordered pair in order to show how a new set can be formed from two given sets.

Let $A = \{1,2,3\}$ and $B = \{4,6\}$. We form from the sets A and B the set of all pairs such that each pair contains an element of A as first element and an element of B as second element. The pairs which are the elements of this new set are (1,4), (1,6), (2,4), (2,6), (3,4), and (3,6). We designate such a set of ordered pairs of A and B by "$A \times B$", read "the Cartesian product of A and B" or simply "A cross B". Thus.

$$A \times B = \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)\}$$

(Note: The set $A \times B$ is named after the mathematician René Descartes who, in the seventeenth century, studies such sets.) Observe that set A contains three elements, set B contains two elements, and the set $A \times B$ contains six elements.

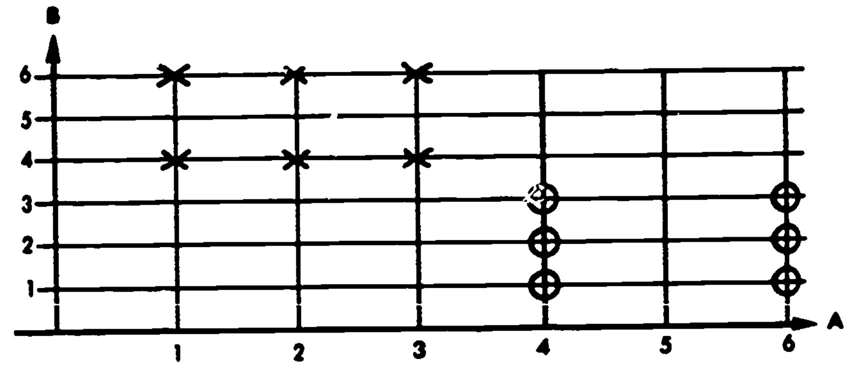
Given the same sets A and B as above we can also form the set $B \times A$. We have

$$B \times A = \{(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)\}$$

We see that if we interchange the order of the pairs in $A \times B$ we obtain $B \times A$. It is important to note that although $B \times A$ also contains six elements we have

$$A \times B \neq B \times A.$$

We can illustrate this inequality by graphing the lattice points associated with each of the Cartesian products. In the graph below we see that elements of $A \times B$ are represented by points denoted by crosses whereas the elements of $B \times A$ are represented by points denoted by circles.



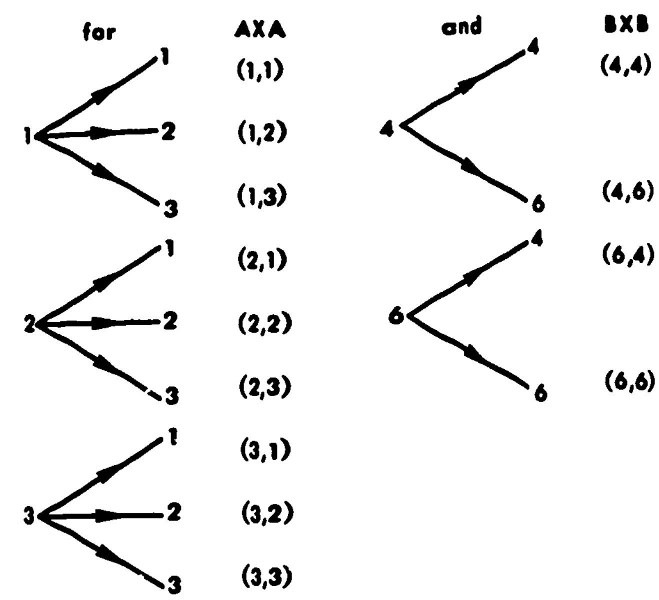
In mathematics we most frequently form the Cartesian product of a set with itself. Thus, if we again used the given sets A and B we obtain

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

and

$$B \times B = \{(4,4), (4,6), (6,4), (6,6)\}$$

We can also use tree diagrams to represent Cartesian products. Thus we would have



The following examples illustrate other instances where we consider the Cartesian product of a set with itslef.

Example 1: Let the set S represent the outcome set of a toss of a single die, that is $S = \{1,2,3,4,5,6\}$. Then $S \times S$ would represent the outcome set for the toss of a pair of dice.

Example 2: Let $T = \{a\}$. Then $T \times T = \{(a,a)\}$. Note that $T \neq T \times T$.

Example 3: Let $\mathbb{Z}$ represent the set of integers. Then $\mathbb{Z} \times \mathbb{Z}$ can be represented by the set of lattice points in the plane.

Example 4: Let W be the set of whole numbers. The operation of addition on W is a mapping which assigns to every element of $W \times W$ a unique element of W called a sum. In symbols $+$

$$W \times W \xrightarrow{+} W$$

We now summarize our ideas about Cartesian product sets with the following definitions:

Definition: The Cartesian product $A \times B$ of two sets A and B is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$.

More compactly we have

Definition: $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

Very often in mathematics we are interested in subsets of Cartesian products. Since the elements of a Cartesian product set are ordered pairs of elements, the elements of non-empty subsets of this Cartesian product set are also ordered pairs. For example, if $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

then $X = \{(1,1), (2,2), (3,3)\}$ and

$Y = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

are proper subsets of $A \times A$. Such subsets as X and Y of AXA are called relations on $A \times A$. It is important to note that a relation is a set of ordered pairs.

We could define the relation X on $A \times A$ by an explicit rule such as $X = \{(a,b) : a \in A, b \in A, a = b\}$ and similarly we could define the relation Y on $A \times A$ by the rule.
$Y = \{(a,b) : a \in A, b \in A, a < b\}$. Relations are usually given by some explicit rule. Note however that

$$T = \{(1,1), (2,1), (3,2)\}$$

is also a relation on $A \times A$ although an explicit rule which defines this relation may not be apparent. Again, a relation is a set of ordered pairs.

If an ordered pair of elements (a, b) is in the relation R then we shall express this by writing

$$(a,b) \in R$$

or by writing

$$aRb$$

We read this latter notation as "a is in the relation R to b". Thus for the relation X we have $(1,1) \Sigma X$, $(2,2) \in X$, $(3,3) \in X$ or, equivalently, 1X1, 2X2, 3X3.

Similarly for the relation Y we have $(1,3) \in Y$ or, equivalently, 1Y3. It may appear strange, at first, to see such statements as "1Y3". However, a familiar example of the "aRb" notation is seen when we consider the relation "equality", denoted by the symbol "=", on the set $W \times W$. If we write "a = b", and which we read as "a equals b", where a and b are whole numbers we always mean that "a" and "b" are different names for the same whole number. Thus we have $1=1$, $2 = 1+1$, $3 = 2+1$, $0 = 1-1$, etc. Let us consider the following examples of relations.

Example 1: Let $D = \{2, 3, 5\}$. We define a relation $\llcorner$ on $D \times D$ as follows: $(a,b) \in \llcorner$ or $a \llcorner b$ if and only if $a \in D$, $b \in D$, and $a < b$. Thus $\llcorner = \{(2,3), (2,5), (3,5)\}$
We could write

$(2,3) \in \llcorner$, $(2,5) \in \llcorner$, and $(3,5) \in \llcorner$

or equivalently

$2 \llcorner 3$, $2 \llcorner 5$, and $3 \llcorner 5$.

(We usually express the above by writing $2 < 3$, $2 < 5$, and $3 < 5$.)

Example 2: Let $A = \{2, 3, 5, 6\}$. We define a relation D on A as follows: aDb if and only if $a \in A$, $b \in A$, and a "divides" b.

Hence, $D = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6)\}$. We could also write

2D2, 2D6, 2D3, 3D6, 5D6, and 6D6.

Observe that $D \subseteq (A \times A)$.

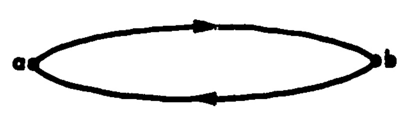
Note: We frequently denote the relation "divides" by the symbol "$|$". Then we would express the above by $2|2$, $2|6$, $3|3$, $3|6$, $5|5$, and $6|6$. The fact that "3 does not divide 5" could be written as $3 \nmid 5$ or $3 \not{D} 5$ or $(3,5) \notin D$.

We could easily graph the relation given in Example 2 after constructing the coordinate diagram of $A \times A$. Instead we shall represent the relation by a different device. A convenient way to designate some relations is by use of arrow diagrams.

If aRb, then we designate two points and label them "a" and "b". Because aRb we direct an arrow from the point labeled "a" to the point labeled "b".

Note that if bRa then the direction of the arrow would be reversed.

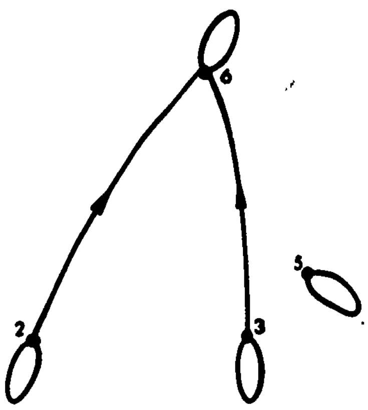
If we have both aRb and bRa then indicate both of these instances by



If it is the case that a R a then we draw a loop at the point labeled "a".

Thus we can draw the following arrow diagram to represent the relation D in Example 2 above:

Observe that an arrow is drawn which connects "2" to "6" because $2|6$ and also an arrow is drawn which connects "3" to "6" because $3|6$. Note that



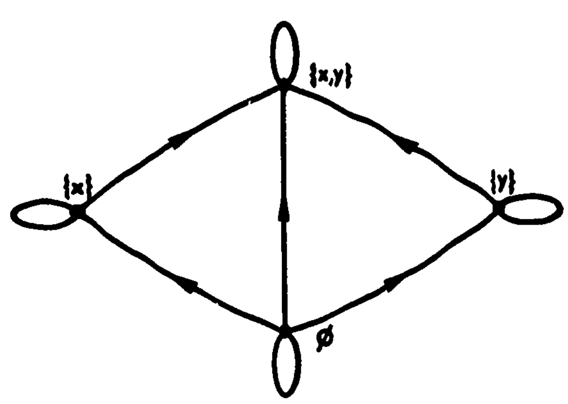
no arrow joins "2" to "5" because 2 does not divide 5 (i.e., $2 \nmid 5$). Note also the loops at each point which indicates that each of the numbers divides itself.

Example 3: Let P be the set of all subsets of the set $\{x, y\}$.

The set P then is given by

$P = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$

Consider the relation "is a subset of" denoted by "$\subseteq$", on the set P. We use an arrow diagram to indicate which elements of P are subsets of each other.



Observe that at each point representing an element of P we have a loop. This is because the elements of P, namely $\emptyset$, $\{x\}$, $\{y\}$, $\{x,y\}$ are sets, and every set is a subset of itself. Also, "$\emptyset$" is connected to "$\{x\}$", "$\{y\}$", and "$\{x,y\}$" because the empty set $\emptyset$ is a subset of every set. Further, both "$\{x\}$" and "$\{y\}$" are connected to "$\{x,y\}$" since $\{x\} \subseteq \{x,y\}$ and $\{y\} \subseteq \{x,y\}$. Do you see that there are nine elements, that is, nine sets of ordered pairs of elements, ir the relation "$\subseteq$" on P?

Example 4: As in Example 3 let P be given by

$P = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$.

The relation "is a proper subset of", denoted by "$\subset$", on the set P is a subset of the relation represented in the arrow diagram above. If the loops are removed from that diagram then we have a representation for "$\subset$" on P.

Example 5: Let Z represent the set of integers.

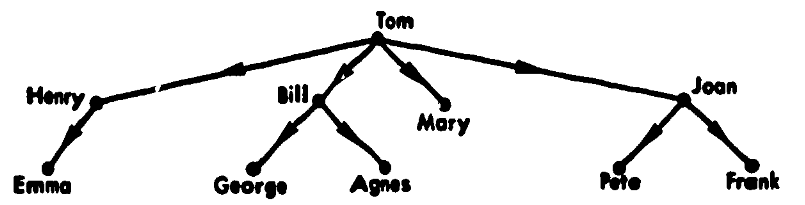
Define the relation S on Z as follows:

aSb if and only if b is the square of A. Thus

$S = \{(0,0), (1,1), (-1,1), (2,4), (3,9), (-3,9), \ldots\}$. Observe that $S \subseteq (Z \times Z)$ (Because we have both 2S4 and -2S4 we see that the relation S is <u>not</u> a mapping of Z into Z. Why?)

Example 6: Let C represent the students in a classroom. Define the relation L on C as follows. Two students x and y are in the relation L on C if and only if x "lives within 1 block of" y. If C represents the students in your class is the relation L on C an empty set?

Example 7: The following arrow diagram shows a simplified family tree.



The above tree indicates that Tom had four children, namely, Henry, Bill, Mary, and Joan. Henry had one daughter, Emma. Bill and Joan each had two children whereas Mary had none. Using first letters to represent people we see that the relation "is a grandfather of" is the set $\{(T,E), (T,G), (T,A), (T,P), (T,F)\}$.

Let us now summarize some ideas associated with the concept of relation. A binary relation (or relation) R from a set A to a set B assigns to each ordered pair (a, b) in $A \times B$ exactly one of the following statements:

(i) "a is related to b", written "aRb"

(ii) "a is not related to b", written "$a \not{R} b$"

A relation R from a set A to a set B is a subset of $A \times B$. Since this is true we see that every relation is a set of ordered pairs, if the relation is non-empty. In mathematics we are most often concerned with a relation R from a set A to the same set A. We say, in this case, that R is a relation on the set A. Here, of course, $R \subseteq (A \times A)$.

8.10 Exercises

1. Using Example 7 in 8.9 list the elements in the following relations: (Note: Use first letters to represent people.)

(a) is a father of

(b) is a brother of

(c) is a grandmother of

(d) is an uncle of

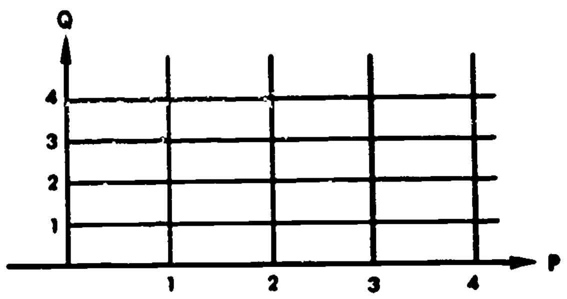
(e) is a sister of

2. Let $P = \{1,2\}$ and $Q = \{2,3,4\}$. Determine the following Cartesian products:

(a) $P \times Q$ (c) $P \times P$

(b) $Q \times P$ (d) $Q \times Q$

3. Copy the coordinate scheme, given below, on your paper. Using Exercise 2 above graph the following Cartesian products using the symbols indicated:



(a) graph $P \times Q$ using crosses ($\times$)

(b) graph $Q \times P$ using circles (o)

(c) graph $P \times P$ using triangles ($\triangle$)

(d) Determine the following:

(1) $(P \times Q) \cap (Q \times P)$

(2) $(P \times P) \cap (Q \times P)$

(3) $(P \times P) \cap (P \times Q)$

(4) $P \times (P \cap Q)$

(5) $(P \times P) \cup (P \times Q)$

(6) $P \times (P \cup Q)$

(e) On the basis of your answers to 3(d) above can you make any conjectures?

4. Let $M = \{1,2\}$, $N = \{2,3\}$, and $P = \{4,5\}$

(a) Determine the following:

(1) $(M \times N) \cup (M \times P)$

(2) $M \times (N \cup P)$

(b) Graph the results found in (a). Can you make an observation?

5. Let $A = \{0, 2, 4\}$ and $B = \{0, 1, 2\}$

Let R be the relation "is greater than," denoted by "$>$", from A to B, i.e., aRb if and only if $a > b$.

(a) Write R as a set of ordered pairs.

(b) Of what set is R a subset?

(c) Explain why or why not 0R2.

(d) Explain why or why not 4R3.

6. Let $B = \{2, 4, 5, 8, 15, 45, 60\}$. Let R be the relation "divides," denoted by "$|$", on the set B, i.e., aRb if and only if $a | b$.

(a) Write R as a set of ordered pairs.

(b) Of what set is R a subset?

(c) Represent the set R by means of an arrow diagram.

(d) Explain how your diagram does or does not indicate the following:

(1) $2|2$ (4) $2|45$

(2) $2|4$ (5) $45|5$

(3) $2|8$ (6) $60|60$

7. (a) Let S be the set of all subsets of the set $\{x\}$. Draw an arrow diagram to represent the relation "is a subset of", denoted by "$\subseteq$", on the set S.

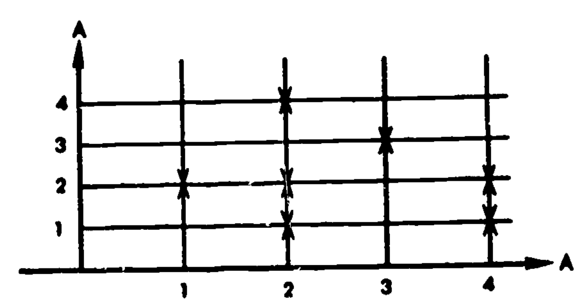
(b) Let T be the set of all subsets of the set $\{x, y, z\}$. Draw an arrow diagram to represent the relation "is a subset of", denoted by "$\subseteq$", on the set T.

8. Let $V = \{1, 2, 3, 4, 5\}$. We define a relation R on V by means of the following table

a \ b	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	5	1	2	4
4	4	3	5	1	2
5	5	4	2	3	1

Prove that for all a, b, and c in V that it is not true that $aR(bRc) = (aRb)Rc$.

9. Let $A = \{1, 2, 3, 4\}$. We define a relation R on A as the set of ordered pairs of numbers designated by crosses ($\times$) in the coordinate diagram of $A \times A$ given below.



(a) Explain why each of the following is true or false:

(1) 1R1 (5) $4\not R3$

(2) 2R2 (6) $4\not R2$

(3) 3R2 (7) $4\not R4$

(4) 2R4 (8) $3\not R3$

(b) Find $\{x : (x, 2) \in R\}$, that is, find all the elements in A which are related to 2.

(c) Find $\{x : 4Rx\}$, that is, find all the elements in A to which 4 is related.

10. (a) Is every relation a mapping? Explain.
    (b) Is every mapping a relation? Explain.
    (c) Let the relation R from A to B be sketched on the coordinate diagram of $A \times B$. What test could one devise in order to determine whether or not R is a mapping of A into B?
11. Research Problem: If set A has m elements and set B has n elements, then how many different relations could we define from A to B? Experiment and write a report of your findings.

8.11 Properties of Relations

Recall Example 2 in 8.9 where $A = \{2, 3, 5, 6\}$ and we defined a relation "divides", denoted by "D" on A. We saw that this relation was a subset of the Cartesian product of set A with itself. In this section we shall consider only relations R which are a subset of the Cartesian product of some set A with itself. That is,

$$R \quad A \times A.$$

Again we shorten this by saying R is a relation on a set A.

Earlier we said that if R is a relation on a set A we could obtain another relation by interchanging the elements (a,b) of R. The relation D in Example 2 of 8.9 was

$D = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6)\}$

The new relation obtained by interchanging the elements of D we shall call the inverse relation of D, and denote it by "D^{-1}". Thus, $D^{-1} = \{(2,2), (6,2), (3,3), (6,3), (5,5), (6,6)\}$

Observe that, here, $D \neq D^{-1}$. Why?

If again $A = \{2, 3, 5, 6\}$, then

$T = \{(2,2), (3,3), (5,5), (6,6)$ is a relation on A.

Observe that, here, $T = T^{-1}$.

The above suggests the following definition:

Definition: Let R be a relation on a set A. The inverse relation of R, denoted by "R^{-1}", consists of exactly those ordered pairs (b,a) such that $(a,b) \in R$. In short,

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

Question 1: If S is the relation on Z given in Example 5 in 8.9 that is

$S = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), (3,9), \ldots\}$, then how can you describe the inverse relation S^{-1} on Z?

Question 2: How can you describe the inverse relation of the relation described in Example 7 of 8.9?

The relation D on set A in Example 2 of 8.9 has a property which we can see by re-examining the arrow diagram of this relation. This is, at every point we find a loop. This is because $2|2$, $3|3$, $5|5$ and $6|6$. Thus for every $a \in A$, we have a D a or $(a,a) \in D$. We describe this situation by saying D is a reflexive relation on A.

Similarly, the relation "$\subseteq$" on the set $P = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$, as given in Example 3 in 8.9 is a reflexive relation on P. Again, the arrow diagram indicates this reflexive property.

The relation "$\subset$" on the same set P, as given in Example 4 in 8.9 is not a reflexive relation on P, because it is not true that for all $a \in P$, we have a a. (In fact, for all $a \in P$, we have $a \not\subset a$. This situation is described by saying "$\subset$" is irreflexive on P.)

Let us make a precise statement concerning this property of reflexivity that a relation on a set may or may not possess.

Definition: Let R be a relation on a set A. R is called a reflexive relation on A if and only if for every $a \in A$, $(a,a) \in R$ or aRa. In other words, R is reflexive on A if and only if every element in A is related to itself.

Question 3: Let S be the relation on Z given in Example 5 in 8.9, that is

$S = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), \ldots\}$. Is S reflexive on Z? Explain.

Question 4: Let $V = \{a, e, i, o, u\}$. Let R be the relation on V given by

$R = \{(a,a), (a,e), (e,e), (e,i), (i,i), (o,o), (u,u)\}$.

Is R reflexive on V? Explain.

Question 5: As in Example 6 of 8.9, let C represent the students in a classroom. Let L denote the relation "lives within 1 block of" on C. Is L reflexive on C? Explain.

Certainly, one of the most basic relations that we encounter with the set of whole numbers W is that of "equality," denoted by "=". Throughout this course we have assumed that for all $x \in W$, $x = x$. In short, we have assumed that "is equal to" is a reflexive relation on the set W.

Another important property of the relation "is equal to" on the set W which we have assumed is the following:

If x, y are whole numbers and $x = y$, then $y = x$. We express this property by saying that "is equal to" is a symmetric relation on the set W.

Again let C represent the students in a classroom and L denote the relation "lives within 1 block of" on C. It is evident that if Bill lives within one block of Jim, then Jim lives within one block of Bill. The relation "lives within one block of" is a symmetric relation on C.

Observe that when discussing whether or not a relation is symmetric on a set we encounter such statements as

If $x = y$, then $y = x$, and

If Bill lives within one block of Jim, then Jim lives within one block of Bill.

Both of the above statements are of the form, "If p, then q." Such statements are called <u>conditional</u> statements and are denoted by

$$p \longrightarrow q$$

The conditional "$p \longrightarrow q$" can also be read as "p implies q." Conditional statements occur frequently, especially in mathematics. Therefore, as your study of mathematics progresses, you will become more familiar with properties of conditional statements. For now, let us note the following:

Remark: The conditional $p \longrightarrow q$ is true unless p is true and q is false. In other words, we do not allow a true statement to imply a false statement.

Another common statement in mathematics is the form "p if and only if q." Such statements are called biconditional statements and are denoted by

$$p \longleftrightarrow q$$

You recall that "if and only if" occurs in many of the definitions that you have seen. If $p \longleftrightarrow q$ we mean that both $p \longrightarrow q$ and $q \longrightarrow p$. Since both "$\longrightarrow$" and "$\longleftrightarrow$" are relations on statements we can determine what properties they possess. We find that "$\longrightarrow$" is reflexive, that is, for all statements p, it is true that $p \longrightarrow p$. In other words, a statement always implies itself. However, "$\longrightarrow$" is not a symmetric relation. It is false that if we know $p \longrightarrow q$, that we can conclude that $q \longrightarrow p$. For example, the following statement is true:

If a triangle has three sides with equal measure, then it has two sides with equal measure.

However, the following statement is not a true statement:

If a triangle has two sides with equal measure, then it has three sides with equal measure.

If we denote "a triangle has three sides with equal measure" by "p" and denote "a triangle has two sides with equal measure" by "q", we see that

$p \longrightarrow q$ does not always imply $q \longrightarrow p$.

Thus "$\longrightarrow$" is not a symmetric relation.

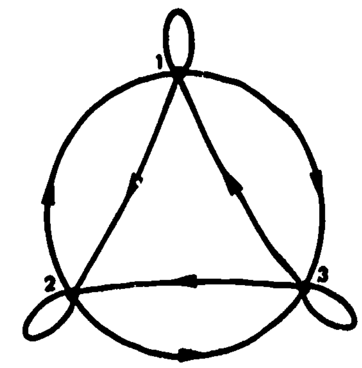
The above examples suggest the following definition:

Definition: Let R be a relation on a set A where a and b are any elements of A. We say R is a symmetric relation on A if and only if aRb implies that bRa.

We can rewrite the above as

R is a symmetric relation on $A \longleftrightarrow (aRb \longrightarrow bRa)$

Example 1: Let $K = \{1, 2, 3\}$. An easily found relation R on K is to consider the Cartesian product $K \times K$. Since $K \times K \subseteq K \times K$, $K \times K$ is a relation R on K. We find that $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,3), (3,4)\}$. Note that $1R2 \longrightarrow 2R1$, $1R3 \longrightarrow 3R1$, etc. If we consider the arrow diagram of R on K

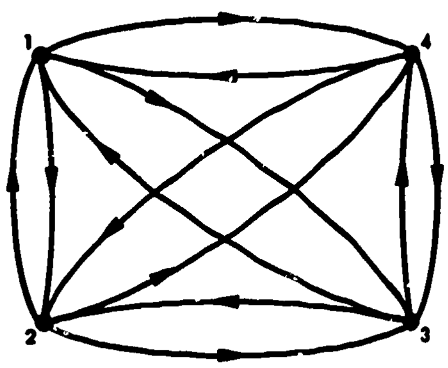


We observe that R is a symmetric relation on K since whenever there is an arrow from a to b there is a corresponding arrow from b to a. Recall that the loops signify R is also a reflexive relation on K.

Example 2: Let $K = \{1, 2, 3\}$ as above and consider the relation R^{-1} on K. Recall $R^{-1} = \{(b,a) : aRb\}$. We find that interchanging the elements in each ordered pair of R gives rise to the same set. Thus $R^{-1} = R$. In Example 1 we saw that R is symmetric on K. Since $R^{-1} = R$ then, of course, R^{-1} is also a symmetric relation on K. Note that the arrow diagram of R^{-1} is the same as the arrow diagram of R. An examination of Example 2 leads us to the following remark:

Remark: If R is a symmetric relation on A, then R^{-1} is a symmetric relation on A.

Example 3: Let $J = \{1, 2, 3, 4\}$. Let us define a relation S on J as follows: If $a, b \in J$ then $aSb \longleftrightarrow a \neq b$. Thus 1S4 because $1 \neq 4$. Also 4S1 because $4 \neq 1$. The arrow diagram for S on J indicates that S is a symmetric relation on J.



Example 4: Let Z be the set of integers. The relation "less than or equal to", denoted by "$\leq$" is not a symmetric relation on Z because for all a, b $\in$ Z, a $\leq$ b does not imply that b $\leq$ a. For example 3 $\leq$ 4 does not imply 4 $\leq$ 3.

The relation described in Example 4, that is "$\leq$" on Z, is not symmetric, however it does illustrate an interesting property. What could you conclude if you were told for two integers a and b that a $\leq$ b and also that b $\leq$ a? The only way that this could be true, would be to have a = b. We describe this situation by saying that the relation "$\leq$" is anti-symmetric on Z. Note that if a $\neq$ b, then possibly a < b or possibly b < a, but never both. In general we have the following definition.

Definition: Let R be a relation on a set A where a and b are any elements of A. We say R is an anti-symmetric relation on A if and only if aRb and bRa implies a = b.

Example 1: Let N be the set of natural numbers and let D be the relation in N defined by "x divides y". D is anti-symmetric on N since x divides y and y divides x implies x = y.

Example 2: Let S be a collection of sets, and let R be the relation on S defined by "A is a subset of B." Then R is anti-symmetric on S since A $\subseteq$ B and B $\subseteq$ A implies A = B.

Question: Can you describe a relation on a set which is not anti-symmetric?

The next property that we shall examine is illustrated by the following: We know for the set W of whole numbers that if a = b and b = c, then a = c. The relation of "equality" is said to be a transitive relation on W. Similarly, if we examine the relation R in Example 2 above we should recall that if A $\subseteq$ B and B $\subseteq$ C, then A $\subseteq$ C. We say that the relation "$\subseteq$" is a transitive relation on S. The general property is given in the following definition.

Definition: Let R be a relation on a set A where a, b, and c are any elements of A. We say that R is a transitive relation on A if and only if aRb and bRc implies aRc.

Example 1: Let Z be the set of integers and let R be the relation on Z defined by "x is less than y." Then R is a transitive relation on R since

if x < y and y < z, then x < z.

In particular,

0 < 7 and 7 < 100 $\longrightarrow$ 0 < 100.

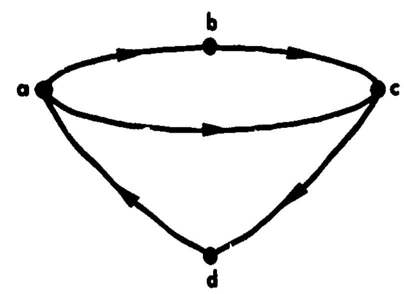
Also,

$-5 < -3$ and $-3 < -1 \longrightarrow -5 < -1$

(Recall that the symbol "$\longrightarrow$" can be read "implies")

Example 2: Let H = {a, b, c, d}. Let us define a relation R on H as follows. R = {(a,b), (b,c), (a,c), (c,d), (d,a)}

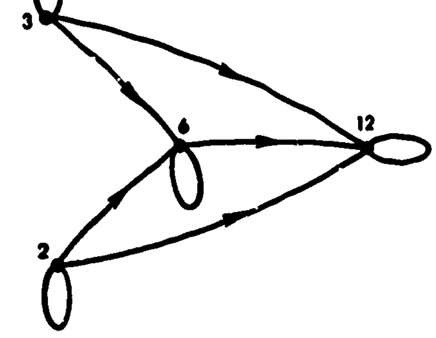
If we examine the arrow diagram of the relation R on H



we see that aRb and bRc $\longrightarrow$ aRc. However,

aRc and cRd does not imply that aRd. The arrow points from d to a. This means dRa and not the required aRd. Since the relation R fails to be transitive for at least one triple of elements of H, we say that R is not transitive on H.

Example 3: Let B = {2, 3, 6, 12} and let D be the relation "divides" on B. The arrow diagram below indicates that 3 | 6 and 6 | 12 $\longrightarrow$ 3 | 12. Also 2 | 6 and 6 | 12 $\longrightarrow$ 2 | 12.



We say that D is a transitive relation on B.

We have pointed out in this section that the important relation of "equality" on the set W satisfies three properties, namely, the reflexive, the symmetric and the transitive properties. That is,

(i) For every whole number a, a = a (Reflexivity)

(ii) For any whole numbers a and b, $a = b \rightarrow b = a$ (Symmetry)

(iii) For any whole numbers a, b, and c, a = b and $b = c \rightarrow a = c$ (Transitivity)

In the next section we shall see that if any relation on a set has these three properties some important results can be derived. Because equality on a set is reflexive, symmetric, and transitive we refer to any other relation on a set which has these properties as an equivalence relation. Thus we have the following definition:

Definition: A relation R on a set A is an equivalence relation if and only if

(1) R is reflexive, that is, for every $a \in A$, aRa.

(2) R is symmetric, that is, for every a and b in a, aRb implies bRa.

(3) R is transitive, that is, for every a, b, and c in A, aRb and bRc implies aRc.

Example 1: Consider the relation "has the same first name as" on the set C of students in a classroom. We must check to see that the requirements in the above definition are satisfied. Let x, y, and z be any students in the class. Then

(i) x has the same first name as x

(ii) if x has the same first name as y, then y has the same first name as x.

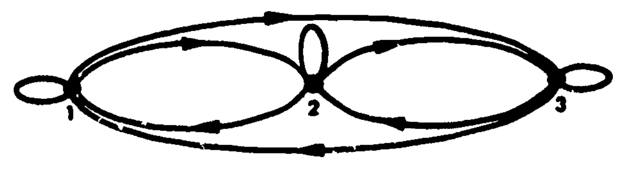
(iii) if x has the same first name as y and y has the same first name as z, then x has the same first name as z.

Since each of the above is true, "has the same first name as" is (i) reflexive, (ii) symmetric, and (iii) transitive and hence is an equivalence relation on C.

Example 2: Consider the relation " $\subseteq$ " on all the subsets of A = {a,b}.

We find that " $\subseteq$ " is reflexive and transitive on A, but since $\{a\} \subseteq \{a,b\}$ does not imply $\{a,b\} \subseteq \{a\}$ we have that the relation is not symmetric on A. Hence it is not an equivalence relation on A.

Example 3: Let K = {1, 2, 3} with a relation defined on it which is illustrated by the arrow diagram below.



Examine this diagram and convince yourself that the relation illustrated is (i) reflexive, (ii) symmetric, and (iii) transitive on K, and hence is an equivalence relation.

8.12 Exercises

1. Let E = {a, b, c} with the following relation R defined on it.

   R = {(a,a), (a,b), (b,c), (b,b), (c,c), (b,a)}

   (a) Explain why R is a relation on E.

   (b) Draw an arrow diagram which represents R on E.

   (c) Explain why or why not R is (1) reflexive, (2) symmetric.

   (d) Write out the ordered pairs in R^{-1}.

   (e) Draw an arrow diagram for R^{-1} on E.

2. Let S be a relation on a set F where F = {1, 2, 3, 4} and S = {(1,1), (1,3), (2,2), (2,3), (2,1), (3,2), (3,3), (3,4), (4,1)}

   (a) Draw an arrow diagram for S.

   (b) Explain why or why not S is

   (1) reflexive
   (2) symmetric
   (3) transitive

3. Each of the following open sentences defines a relation on the set W of whole numbers. Determine if each is or is not a reflexive relation on W.

   (a) "a is less than or equal to b"

   (b) "a + b = 8"

   (c) "a divides b"

   (d) "a is greater than b"

   (e) "a is equal to b"

   (f) "the square of a is b"

   (g) "a − b is divisible by 5"

4. Using the open sentences in Exercise 3 determine if each is or is not a symmetric relation on W

5. Using the open sentences in Exercises 3 determine if each is or is not a transitive relation on W.

6. Which of the relations, in Exercise 3, if any, are equivalence relations?

7. (a) When is a relation R in a set A not reflexive?

   (b) When is a relation R on a set A not symmetric?

   (c) When is a relation R on a set A not transitive?

8. Let $A = \{1, 2, 3\}$. Consider the following relations on A.

   $R_1 = \{(1,1), (1,2), (1,3), (2,1), (2,3)\}$

   $R_2 = \{(1,1), (2,3), (3,2), (1,2), (3,1)\}$

   $R_3 = \{(1,2), (2,3), (1,3)\}$

   $R_4 = \{(1,1)\}$

   $R_5 = A \times A$

   Determine whether or not each of these relations is

   (a) reflexive

   (b) symmetric

   (c) transitive

9. Examine the relation "is a brother of" for a set of people with respect to

   (a) reflexivity

   (b) symmetry

   (c) transitivity

10. Let $A = \{a, b, c\}$. Consider the following relations on A.

    $R_1 = \{(a,a), (b,a), (b,b), (c,b), (b,c)\}$

    $R_2 = \{(a,a)\}$

    $R_3 = \{(a,b)\}$

    $R_4 = \{(a,a), (b,c), (c,b)\}$

    $R_5 = A \times A$

    Determine whether or not each of the above relations is anti-symmetric.

11. Let L be a set of lines in the plane and let P be the relation on L defined by "ℓ_1 is parallel to ℓ_2".

    Determine whether or not P is (a) reflexive, (b) symmetric, (c) transitive, (d) anti-symmetric, (e) an equivalence relation. (Assume a line is parallel to itself)

12. Let S be the collection of subsets of $\{x, y, z\}$. If A and B are elements of S the following are relations on S:

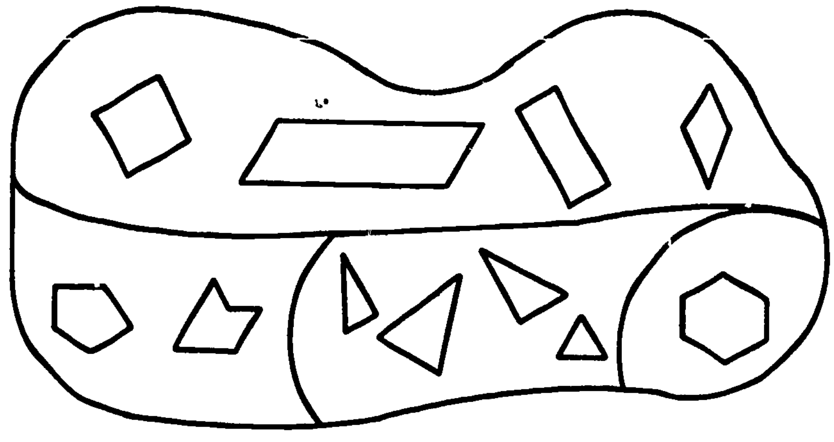
    (i) "$A \subseteq B$" (iii) "A is disjoint from B"

    (ii) "$A \subset B$" (iv) "A is not equal to B"

    Determine if the above relations on S are (a) reflexive, (b) symmetric, (c) transitive and (d) anti-symmetric.

8.13 Partitions

Examine the drawing below in which we have drawn a closed curve about a set of eleven geometric figures. Let us designate this set of figures as G.



We see that not all of the figures have the same number of sides. In fact we find there are four 3-sided figures (i.e., 4 triangles), four 4-sided figures (i.e., 4 quadrilaterals), two 5-sided figures (i.e. 5 pentagons) and one 6-sided figure (i.e., 1 hexagon).

Next we define a relation R on the set G as follows: If x and y are any elements of G we say that xRy if and only if x and y have the same number of sides.

Thus any two triangles in G are in the relation R to each other whereas a triangle and a quadrilateral are not in the relation R to each other.

Because every geometric figure in G has the same number of sides as itself, we have that R is reflexive on G. If x has the same number of sides as y, then y has the same number of sides as x. Hence, R is symmetric on G. Also if x has the same number of sides as y and y has the same number of sides as z, then x has the same number of sides as z. Thus R is transitive on G. From the above we conclude that R is an equivalence relation on G.

Let us now examine the effect of the equivalence relation R on the set G. It is important to note that the relation R effects a separation of the elements of G into subsets. Each of these subsets contains exactly those geometric figures which have the same number of sides (See how this is indicated in the drawing above). Let us designate these subsets of G as T (the set of triangles), Q (the set of quadrilaterals), P (the set of pentagons), and H (the set of hexagons). The collection of subsets of G

$$\{T, Q, P, H\}$$

produced by the equivalence relation R on G we call a partition of G.

The subsets which form the partition of G have two important properties which we next observe.

Observation 1: The union of the subset T, Q, P, and H of G is the set G. That is

$$T \cup Q \cup P \cup H = G$$

Observation 2: The subsets T, Q, P, and H of G are pairwise disjoint. This means that if we consider any two distinct subsets their intersection is the empty set. We see that this is true because

$$T \cap Q = \emptyset, \; T \cap P = \emptyset, \; T \cap H = \emptyset$$

$$Q \cap P = \emptyset, \; Q \cap H = \emptyset, \text{ and } P \cap H = \emptyset.$$

It is no accident that R effected a partition of G into pairwise disjoint subsets whose union is G. It turns out that we obtained such a partition of G because R is an equivalence relation on G. The most significant property of an equivalence relation on a set is that it always partitions the set into pairwise disjoint subsets whose union is the given set.

We could also say that an equivalence relation R on a set A partitions the set by putting those elements which are related to each other in the same subset of A. Each of these subsets is called an equivalence class. In the example above T, Q, P, and H are equivalence classes. The following examples will illustrate many of the ideas examined above.

Example 1: We shall define a relation R on the set Z of integers as follows: Let x and y be any two integers. We say xRy if and only if both x and y are even or both x and y are odd. Thus $^-3R7$ but $^-3\not R8$. The relation R is an equivalence relation on Z. (Prove this.) Moreover the relation R establishes two subsets of Z:

$E = \{x : x \in Z, \text{ x is even}\}$ and

$O = \{x : x \in Z, \text{ x is odd}\}$

Every integer in Z is either an element in E or an element in O, but never an element in both E and O.

(i) $E \cup O = Z$ and

(ii) $E \cap O = \emptyset$

The equivalence relation R on Z effects a partition on Z. This partition is $\{E, O\}$. E and O are equivalence classes in this partition.

Example 2: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We define a relation R on A as follows: Let a and b be any elements of A. We say aRb if and only if a and b have the same remainder when they are divided by 4. It is easy to see that R is an equivalence relation on A which determines the following subsets of A:

$B_1 = \{1,5,9\}$, $B_2 = \{2,6,10\}$, $B_3 = \{3,7,11\}$, $B_4 = \{4,8,12\}$.

We note that $B_1 \cup B_2 \cup B_3 \cup B_4 = A$ and also that $B_1 \cap B_2 = \emptyset$, $B_1 \cap B_3 = \emptyset$, $B_1 \cap B_4 = \emptyset$, $B_2 \cap B_3 = \emptyset$, $B_2 \cap B_4 = \emptyset$, and $B_3 \cap B_4 = \emptyset$

Thus R effects the partition $\{B_1, B_2, B_3, B_4\}$ on A.

Example 3: Let C be the set of students in a class. It is clear that the relation "has the same first name as" is an equivalence relation on C. Further, this relation partitions C into equivalence classes. (Examine your own class.) It could happen that every student had a different name. If in such a class there are twenty students we find that the equivalence relation still partitions the set. Here each equivalence class would have in it a single element. Thus, the partition would be a set having twenty equivalence classes as elements.

Example 4: Let $A = \{a,b,c\}$. We find that there are five different possible partitions of A. These are,

(i) $\{ \{a, b, c\} \}$

(ii) $\{ \{a\}, \{b, c\} \}$

(iii) $\{ \{b\}, \{a, c\} \}$

(iv) $\{ \{c\}, \{a, b\} \}$

(v) $\{ \{a\}, \{b\}, \{c\} \}$

Each of the five sets above is a partition of A. Again we see that the elements of a partition are sets. In (ii) the elements that make up the partition $\{ \{a\}, \{b, c\} \}$ are the equivalence classes $\{a\}$ and $\{b, c\}$. We have that $\{a\} \cup \{b, c\} = \{a, b, c\} = A$. Also $\{a\} \cup \{b, c\} = \emptyset$.

Similar statements are possible for (i), (iii), (iv), and (v).

You have already encountered, in earlier chapters, examples of equivalence relations inducing a partition on a set. In Chapter 4 you constructed the set of integers. Recall that you were shown that each integer in Z is a set of ordered pairs (a, b) of whole numbers. Such integers as

$$+4 = \{ (0,4), (1,5), (2,6), (3,7), \ldots \}$$

$$0 = \{ (0,0), (1,1), (2,2), (3,3), \ldots \}$$

$$-3 = \{ (3,0), (4,1), (5,2), (6,3), \ldots \}$$

illustrate that every integer is a set of ordered pairs of whole numbers. Using the language of this chapter we can now say that the set Z of integers is a partition of the set $W \times W$. Each integer is an equivalence class in this partition. Observe that the union of all integers is $W \times W$ and the intersection of any pair of distinct integers is the empty set. Do you recall the relation R which effected this partition of $W \times W$ into the equivalence classes called integers?

Using some ideas of this chapter we shall again define that relation. Let (a,b) and (c,d) be any elements of $W \times W$. We say that (a,b)R(c,d) if and only if a+d = b+c. Thus (1,5)R(3,7) because 1+7 = 5+3. If two elements of $W \times W$ are in the relation R to each other they belong to the same equivalence class. We see that this is true here since $(1,5) \in +4$ and $(3,7) \in +4$. It can be shown that the relation R is an equivalence relation on $W \times W$. But we will leave this for an exercise.

We are seldom interested in a set unless some relation or operation has been defined on the set. In this section we have seen that defining an equivalence relation R on a set A yields a partition of A into equivalence classes. We might say that the relation R on A gives a structure to the set A. Of course different relations defined on A yield different structures. We shall next consider another important way to structure a set which deals with the frequently encountered idea of order.

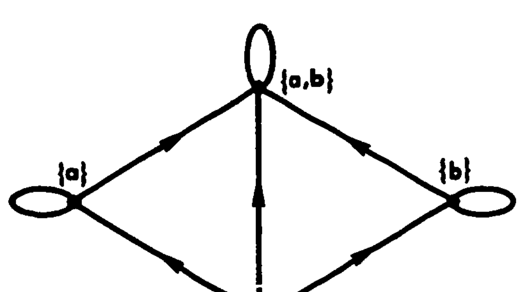
A partial order on a set A is a relation on A which is

(1) reflexive, i.e., for every $a \in A$, aRa

(2) anti-symmetric, i.e., for all a, $b \in R$, aRb and bRa implies a = b.

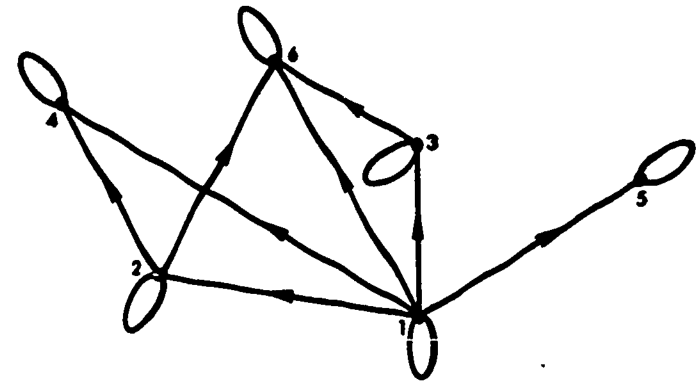
(3) transitive, i.e., for all a, b, $c \in R$, aRb and bRc implies aRc.

Example 1: Let S be the collection of all subsets of {a, b}. The relation "$\subseteq$" defined on S is a partial order on S. Why? This relation is illustrated by the arrow diagram below.



Example 2: Let A = {1, 2, 3, 4, 5, 6}.

The relation "divides" defined on A is a partial order on A. Explain. The arrow diagram below illustrates this relation.



8.14 Exercises

1. Let A = {1, 2, 3, 4, 5, 6}. Explain why each of the following is or is not a partition of A.
   (a) { {1,2}, {5,6,3} }
   (b) { {1,2}, {3}, {4,5}, {6,2} }
   (c) { {1,3,5}, {2,4,6} }
   (d) { {1}, {2}, {3}, {4}, {5}, {6} }
   (e) { {1}, {6,4}, {3,5,2} }
   (f) { {1,2,3,4}, {4,5,6} }
   (g) { {1,2,3,4,5,6} }
   (h) { {1,2}, {3,4} }
2. Find all the partitions of {a,b}.
3. Explain why "<" defined on W does not partition W.
4. Let R be an equivalence relation on A. If we assume that cRa and cRb why can we conclude that aRb?
5. Find all the partitions of {1,2,3,4}.
6. (a) Use an arrow diagram to illustrate the relation "divides" on the set E = {1, 2, 3, 5, 6, 10, 15, 30}. Is this relation a partial ordering on E?
   (b) Explain why or why not the relation "<" is a partial ordering on the set W.
7. Consider the following relations defined on the set P of people in the United States:
   R_1: "lives in the same state as"
   R_2: "lives within 1 mile of"
   R_3: "is the father of"
   R_4: "is a member of the same political party as"
   R_5: "has the same I.Q. as"
   (a) Determine which of the above are equivalence relations on P.

(b) Describe the equivalence classes in the partitions effected by the relations in (a) which are equivalence relations on P.

8. In Section 8.13 we defined a relation R on $W \times W$ as follows: (a,b)R(c,d) if and only if $a+d = b+c$. Prove that R is an equivalence relation.

9. Research Problem: Let R be an equivalence relation on A. For every $a \in A$, let

$$B_a = \{x : xRa\}$$

Prove that R effects a partition on A.

8.15 Summary

In this chapter you have encountered some of the most basic terms used in the study of mathematics. Terms such as set, relation, partition, etc. will become, in time, part of your basic vocabulary.

With respect to sets you should be able to give a clear and complete description of the following terms:

set equality, subset, proper subset, universal set, union, intersection, empty set, complement, disjoint sets, product set.

With respect to relations you should understand what is meant by the following terms:

relation, inverse relation, reflexivity, irreflexivity, symmetry, transitivity, anti-symmetry, equivalence relation, partition, partial order.

Also you should be aware of the tools we have used in our study. These tools include:

set notation, Venn diagrams, membership tables, arrow diagrams, the conditional relation "$\longrightarrow$".

At this time you should review, for yourself, the meanings of the above terms. Restudy any terms whose meanings are not clear to you.

8.16 Review Exercises

1. Let S be a universal set where

$S = \{p, q, r, s, t, u, v\}$

Let $A = \{p, q, r, s\}$, $B = \{t, u, v\}$, $C = \{p, r, t, v\}$, $D = \{t\}$

(a) Determine the following sets:

(1) $A \cup B$	(5) $B \cup C$	(9) $D \cup D$
(2) $A \cap B$	(6) $B \cap C$	(10) $B \cap B$
(3) $A \cup C$	(7) $A \cup D$	(11) $S \cap D$
(4) $A \cap C$	(8) $A \cap D$	(12) $S \cup D$

(b) Find the complement of each of the following sets:

(1) A	(3) C	(5) $A \cup D$
(2) B	(4) $A \cap C$	(6) $A \cup B$

(c) Which of the sets A, B, C, D are

(1) subsets of each other?

(2) proper subsets of each other?

(3) pairwise disjoint?

2. Write three statements that are true of every set A.

3. Let $B = \{x : x \in W, x \text{ is even}, x < 3\}$.

(a) Rewrite set B by listing its elements

(b) List all the subsets of B.

(c) List all the proper non-empty subsets of B.

(d) Determine $B \times B$.

(e) Is $\{(0,0), (0,1)\}$ a relation on B?

(f) Is $\{(0,0), (0,2)\}$ a relation on B?

(g) Draw an arrow diagram for $B \times B$

4. Let $V = \{0, 1, 2, 3\}$. Let R be a relation on V defined as follows:

$R = \{(0,0), (0,1), (1,0), (1,2), (2,1), (2,2), (0,2), (3,3)\}$

(a) Draw the arrow diagram for R on V

(b) Is R an equivalence relation on V?

(c) What would occur if we defined a new relation S on V where

$$S = R \cup \{(1,1), (2,0)\}?$$

5. (a) Prove or disprove that $(A \cap B) \cup (A \cap \overline{B}) = A$

(b) $(A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B) = ?$

6. Give an example of a relation R on a set A which is

(a) reflexive, anti-symmetric, transitive

(b) reflexive, symmetric, transitive

(c) irreflexive, not symmetric, not transitive

(d) irreflexive, not symmetric, transitive

(e) irreflexive, symmetric, transitive

7. Determine which of the following are true

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$

(b) If $A \subset B$, then $\overline{B} \subset \overline{A}$

8. Let $D = \{2, 4, 6, 8, 10, 12\}$

Explain why the following are or are not partitions of D:

(a) $\{\{2,4\}, \{6,10\}, \{4,12\}, \{8\}\}$

(b) $\{\{2,4,6\}, \{8,10\}, \{12\}$

(c) $\{\{2\}, \{6,12\}, \{4,10\}$

9. Let R be a relation on set A. Determine if the following is true: R is symmetric on A if and only if $R = R^{-1}$.

10. Let $B \Delta C = B \cap \overline{C}$. Prove or disprove that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$